Topology Qualifying Exam January 2022

Instructions. Please do each of the problems below, justifying your work to the best of your ability in the allotted time. When you have finished, please write and sign the honor code somewhere on your exam.

"On my honor, I have neither given nor received any unauthorized aid on this exam."

then email it to Chris at cjil12@rice.edu.

1. Let $S$ be a compact, connected, orientable surface of genus 3.
   (a) Which compact, connected, orientable surfaces are covering spaces of $S$?
   (b) How many regular (i.e. normal), connected 7–sheeted covers of $S$ are there, up to isomorphism?

2. Let $G$ be a finitely generated abelian group. Construct a (path-connected) topological space $X_G$ with $\pi_1(G) \cong G$. You must show all your work. You may use the classification of finitely generated abelian groups, which implies $G$ has the form $\mathbb{Z}^k \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$, for some integers $k, p_1, \ldots, p_n$.

3. Consider $X$, the wedge of two circles, viewed as a 1-complex with oriented edges (1-cells) labeled $a$ and $b$, as shown below.

$$a \quad \begin{tikzpicture}[baseline=-0.5ex]
    \draw[thick,->] (0,0) -- (1,0) node[below] {$x_0$} node[anchor=west] {$b$};
    \draw[thick,->] (1,0) -- (2,0) node[below] {$a$};
\end{tikzpicture}$$

Let $f: X \to X$ be a map which sends $x_0$ to itself, sends the oriented edge $a$ to the edge-path $aba$ and the sends the edge $b$ to the edge-path $bab$. Let $M_f = X \times [0,1]/(x,1) \sim (f(x),0)$ be the mapping torus of $f$; that is, the quotient space of $X \times [0,1]$ where every point $(x,1)$ is identified to the point $(f(x),0)$.
   (a) Compute $H_p(M_f)$ for all $p$.
   (b) Compute $H^p(M_f; \mathbb{Q})$ for all $p$, using the Universal Coefficient Theorem.

4. Let $M$ be a compact, connected, orientable 4–dimensional manifold without boundary such that $\pi_1(M) \cong \mathbb{Z}_{15}$ and $H_2(M; \mathbb{Q}) \cong \mathbb{Q}^2$. Let $N$ be a connected 3–fold covering space of $M$.
   (a) Calculate $\pi_1(N)$.
   (b) Calculate $H_p(N; \mathbb{Z})$ for each $p$
   (c) Calculate $\chi(M)$.
   (d) Calculate $H_p(N; \mathbb{Z})$ for each $p$
   (e) Prove that $N$ admits no CW structure without 3–cells.

5. (a) Prove that every map $f : S^4 \to \mathbb{C}P^2$ has degree 0.
   (b) Suppose $M$ is a closed, oriented manifold of dimension 4. Show there is a degree 1 map $f : M \to S^4$. You will need to explicitly describe this continuous map. (Note: Part (b) holds in all dimensions, not just dimension 4.)

6. Let $F: \mathbb{R}^3 \to \mathbb{R}$ be the map given by $F(x, y, z) = z - x^2 - y^2$ and let $f: S^2 \to \mathbb{R}$ be the restriction to the sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. (Alternatively, if $i: S^2 \to \mathbb{R}^3$ is inclusion, then $f = F \circ i$.) Prove that 0 is a regular value of $f$. 