RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2022

This is a 4 hour, closed book, closed notes exam. **Justify all of your work** as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Suppose S is a closed, connected, orientable surface of genus 2 (without boundary) and suppose that $p\colon S' \to S$ is a connected, d–sheeted covering space, with $1 < d < \infty$. What is the genus of S'?

2. For $i = 1, 2$, let $V_i = S^1 \times D^2$ be a solid torus with boundary $\partial V_i = S^1 \times \partial D^2 = S^1 \times S^1$ and let $f\colon \partial V_1 \to \partial V_2$ be the homeomorphism defined by $f(x, y) = (x+2y, y)$ (viewing $S^1 = \mathbb{R}/\mathbb{Z}$). Let $L = V_1 \cup_f V_2$, the space obtained from $V_1 \sqcup V_2$ by identifying ∂V_1 with ∂V_2 via f; that is, L is the quotient space of $V_1 \sqcup V_2$ by the equivalence relation generated by $(x, y) \sim f(x, y)$ for all $(x, y) \in \partial V_1$. Compute $\pi_1(L)$.

3. Let X be the wedge product of a circle and a 2–dimensional torus; that is, X is the quotient space obtained by identifying a point on S^1 with a point on the torus $T^2 = S^1 \times S^1$.

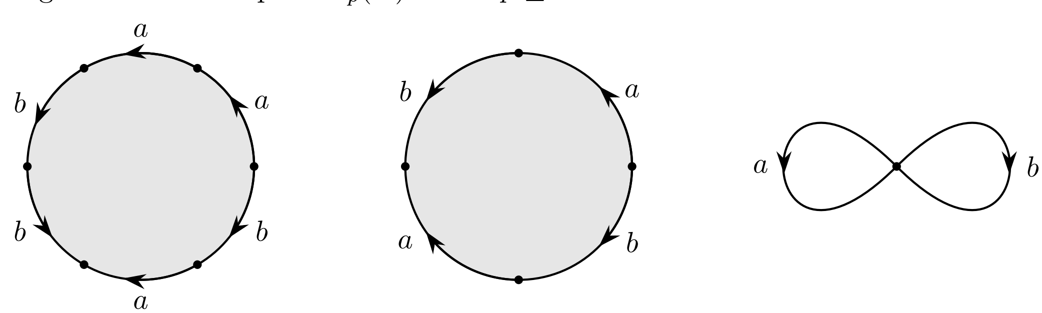
   (a) Compute $\pi_1(X)$.

   (b) How many connected, 2–sheeted covers of X are there, up to isomorphism?

   (c) Is there a regular, connected cover of X with covering group isomorphic to F_2, the free group of rank 2?

   (d) Is there a regular, connected cover of X with covering group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$?

4. Let Y be the CW complex with one 0-cell, two 1-cells, and two 2-cells attached as indicated in the figure below. Compute $H_p(Y)$ for all $p \geq 0$.



5. Let $U \subset \mathbb{R}^4$ be an open subset with $S^2 \times D^2 \cong U$ and $\Sigma \subset U \subset \mathbb{R}^4$ the image of $S^2 \times \{0\}$ under the homeomorphism. Set $X = \mathbb{R}^4 \setminus \Sigma$ and compute $H_p(X)$ for all $p \geq 0$.

6. Let M be a closed, connected, orientable 4–manifold with $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and second betti number $b_2(M) = 0$. Compute $H_p(M)$ and $H^p(M)$ for all $p \geq 0$.