RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2021

This is a 4 hour, closed book, closed notes exam. *Justify all of your work*, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let X be a topological space, $A \subset X$ a subspace, and $r\colon X \to A$ a retraction (i.e. $r(x) = x$ for all $x \in A$). Suppose $r_*\colon \pi_1(X) \to \pi_1(A)$ is injective and that X is homotopy equivalent to a circle, S^1. Prove that A is homotopy equivalent to S^1.

2. Prove that
$$W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 10,\ 4x^2 + 9y^2 - z^2 = 1\}$$
is a smooth, 1–dimensional submanifold of $\mathbb{R}^3$.

3. Let S be the connect sum of a torus and projective plane. List all compact, connected surfaces (without boundary) that **cannot** appear as finite sheeted covering spaces of S. (Your justification should discuss the surfaces on your list and those *not* on your list.)

4. Let X be the wedge product of the projective plane and a circle (that is, the space obtained by gluing these two together at a point).
   (a) Compute $\pi_1(X)$.
   (b) Prove that for every $n \geq 2$, there is a connected, regular covering space of X with covering group isomorphic to S_n, the symmetric group on n elements.
   (c) Compute $H_p(X)$ for all p.

5. Let X be a closed, connected 4-dimensional manifold with $\pi_1(X) \cong 1$ and second Betti number $b_2(X) = m$ for some integer $m \geq 0$.
   (a) Prove that X is orientable.
   (b) Compute $H_p(X)$ and $H^p(X)$ for all p.
   (c) Give an example of a ring R that cannot be isomorphic to the cohomology ring of X, $H^*(X)$ (just as a ring, not a graded ring).

6. (a) Describe a cellular decomposition of $\mathbb{C}P^\infty$, including the attaching maps.
   (b) Use this to compute $H^p(\mathbb{C}P^\infty)$ for all p.
   (c) Prove that the cohomology ring of $\mathbb{C}P^\infty$ is a polynomial ring with one variable and integer coefficients, i.e. $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$.