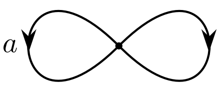
RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2023

This is a 4 hour, closed book, closed notes exam. **Justify all of your work** as much as time allows. **Only turn in solutions to 6 of the 9 problems**. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Prove that a Möbius band admits no retraction onto its boundary circle.
2. Let $f, g\colon S^1 \to S^1$ be given by $f(e^{i\theta}) = e^{10i\theta}$ and $g(e^{i\theta}) = e^{13i\theta}$. Construct Y from the 3–torus, $T^3 = S^1 \times S^1 \times S^1$, by attaching two 2–disks, D^2, via maps $F, G\colon \partial D^2 = S^1 \to T^3$, given by $F(z) = (f(z), x, x)$ and $G(z) = (x, g(z), x)$, for some fixed $x \in S^1$.
   (a) Compute $\pi_1(Y)$.
   (b) How many isomorphism classes of connected 5–sheeted covers of Y are there?
3. Let X be a wedge of two circles as shown to the right. Construct each of the following:[1]
   

   (a) Two connected, regular 4–sheeted covers of X with non-isomorphic covering groups.
   (b) An irregular, connected 4–sheeted cover of X.
4. Let W be the *double* of a solid torus $V = D^2 \times S^1$ over its boundary. Formally, W is the quotient of the disjoint union of two copies of the solid torus, $W = V \times \{1, 2\}/\sim$, where $(x, 1) \sim (x, 2)$ for all $x \in \partial V = \partial D^2 \times S^1$. Compute $H_n(W, \mathbb{Z})$ for all $n \geq 0$.
5. Let X be a closed, connected, orientable 4–manifold with $\pi_1(X) \cong \mathbb{Z}_{15}$ and $H_2(X, \mathbb{Q}) \cong \mathbb{Q}^2$, and let E be a connected 3–sheeted cover of X. Calculate each of the following.
   (a) $H_p(X, \mathbb{Z})$ for all $p \geq 0$, (b) $\chi(X)$, (c) $\pi_1(E)$, (d) $H_p(E, \mathbb{Z})$ for all $p \geq 0$.
6. Suppose $f\colon M \to N$ is a map between closed, connected, oriented n–manifolds. The *degree* of f is the integer, $\deg(f)$, so that $f_*([M]) = \deg(f)[N]$, where $[M] \in H_n(M, \mathbb{Z})$, $[N] \in H_n(N, \mathbb{Z})$ are the fundamental classes. Prove that if $\deg(f) = 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is surjective. **Hint**: *You may use, without proof, the fact that a d–sheeted connected cover of a closed, connected, oriented manifold can be oriented so that the covering map has degree d.*
7. Let M be a nonempty smooth closed manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$. (A submersion is a smooth map whose differential is surjective.)
8. Suppose $F : M \to N$ and $G : N \to P$ are smooth maps between smooth manifolds, and G is transverse to an embedded submanifold $X \subset P$. Show that F is transverse to the submanifold $G^{-1}(X)$ if and only if $G \circ F$ is transverse to X.
9. Let Δ be the distribution on $\mathbb{R}^3$ defined by the form $\omega = dx + zdy + ydz$, that is ω generates the ideal of forms vanishing on Δ.
   (a) Use ω to decide whether or not Δ is integrable.
   (b) Find vector fields generating Δ and use them to give an alternate proof of (a).

[1] Draw pictures of the covers as 4–valent graphs with oriented edges labeled a & b, and include written justification.