RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2019

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

You can assume all spaces are path-connected.

1. Let F be a free group of rank n and let S be a subgroup of F of finite index d.
   (a) Prove that S is a free group.
   (b) Calculate the rank (as a free group) of S in terms of n and d.

2. By definition, a topological group is a set G with both a topology and a group structure ($G \times G \xrightarrow{*} G$), such that the map $G \to G$ sending x to x^{-1} and the map $G \times G \to G$ sending (x, y) to $x * y$ are both continuous. Let $1 \in G$ denote the identity of this topological group G. Show that $\pi_1(G, 1)$ is abelian.

   Hint: Note that $f \cdot g = (f \cdot c_1) * (c_1 \cdot g)$ where c_1 is the constant map based at 1. Here $\cdot$ is the multiplication (concatenation) in π_1 and $a * b$ is defined as $(a * b)(t) = a(t) * b(t)$ for all t.

3. Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. That is,
   $$S(X) = X \times [0, 1]/\sim$$
   where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. Describe the relation between the cohomology groups of X and $S(X)$.

4. Let X be a compact, connected, orientable 4-dimensional manifold without boundary such that $\pi_1(X) \cong \mathbb{Z}_{15}$ and $H_2(X; \mathbb{Q}) \cong \mathbb{Q}^2$. Let E be a connected 3-fold covering space of X.
   (a) Calculate $\pi_1(E)$.
   (b) Calculate $H_p(X; \mathbb{Z})$ for each p.
   (c) Calculate $\chi(X)$.
   (d) Calculate $H_p(E; \mathbb{Z})$ for each p.
   (e) Prove that E admits no CW decomposition without 3-cells.

5. Let X be a closed (compact and boundaryless), oriented 4-manifold with $\beta_2(X) \neq 0$. Prove that any continuous map $f : S^4 \to X$ has degree equal to 0.

6. Prove that the tangent bundle to S^1 is diffeomorphic to $S^1 \times \mathbb{R}$.