Analysis Qualifying Exam, January 2022

Please put your name on your solutions, use different 8 1/2×11 in. sheets for different problems, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

We denote Lebesgue measure on $\mathbb{R}$ by m.

1. Let $\{A_n\}$ be a sequence of measurable sets in the interval $[0,1]$ with $m(A_n) \geq \epsilon$ for some $\epsilon > 0$.
   (a) Show that there exists a point in infinitely many such A_n.
   (b) Does there exist a point which is in all but finitely many A_n?

2. Let f be a function that is analytic in some open set containing $\{z \in \mathbb{C} \mid |z| \leq 1\}$. Suppose that
$$|f(z) - z| < |z|$$
on the unit circle.
   (a) Show that $|f'(1/2)| \leq 8$.
   (b) How many zeros does f have inside the unit circle?

3. Fix $c \in (0,1)$. If a Borel set $E \subset \mathbb{R}$ obeys
$$m(E \cap (a,b)) \leq c(b-a)$$
for all $a, b \in \mathbb{R}$ with $a < b$, prove that $m(E) = 0$.

4. Let f be an analytic function on the unit disk $\mathbb{D}$ with $f(0) = f'(0) = 0$. Prove that
$$g(z) = \sum_{n=1}^{\infty} f(z/n)$$
defines an analytic function on $\mathbb{D}$. Prove that $g(z) = cf(z)$ for some constant $c \in \mathbb{C}$ if and only if $f(z) = \alpha z^k$ for some $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$.

5. Consider the initial value problem
$$y'' = 2y' \tan x + y^2 + 2xy + x^2, \qquad y(0) = 1, \qquad y'(0) = 0$$
defined near the origin.
(a) Show that there is an $x_0 > 0$ at which the solution has $y'(x_0) = 2$.
(b) Show that if $p(x)$ is a polynomial of degree 4, then there exists $s > 0$ so that for the solution to
$$y'' = 2y' \tan x + y^2 + 2xy + x^2 + sp(x), \qquad y(0) = 1, \qquad y'(0) = 0$$
there is a point x_1 for which $y'(x_1) = 2$.

6. Let f be an entire function which satisfies $f(1) = 1$ and $|f(z)| = 1$ if $|z| = 1$. Show that $f(z) = z^k$ for some integer $k \geq 0$.