Analysis Exam, January 2023

Please put your name on your solutions, use different 8 1/2×11 in. sheets for different problems, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. Show that there is no function f that is holomorphic near $0 \in \mathbb{C}$ and satisfies

$$f\left(\frac{1}{n^2}\right) = \frac{n^2-1}{n^5}$$

for all large $n \in \mathbb{N}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Prove that

$$\lim_{a \to +\infty} \int_{-\infty}^{+\infty} f(x) \sin(ax)\, dx = 0.$$

(Or, e.g., with e^{-iax} in place of $\sin(ax)$.)

3. Suppose that a function f is analytic and bounded on the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Prove that f is uniformly continuous on the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$.

4. Let μ be a finite Borel measure on $\mathbb{R}^n$. Let $\bar{B}(x, r)$ denote the closed ball of radius r centered at x and $\partial B(x, r)$ denote its boundary.

   (a) Prove that if $x_n \to x$, then $\mu(\bar{B}(x, r)) \geq \limsup_{n\to\infty} \mu(\bar{B}(x_n, r))$.

   (b) Prove that the map $x \mapsto \mu(\bar{B}(x, r))$ is continuous at x if and only if $\mu(\partial B(x, r)) = 0$.

5. Show that there is a function $f(z)$, analytic in an open neighborhood U of $z = 0$, such that

$$f(z)^{10} = \frac{1}{\cos(z^5 + 2z^7)} - 1$$

for all $z \in U$.

6. Assume $f \in C_c^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} e^{-tx^2} f(x) dx = 0$$

for any $t \geq 0$. Show that $f(x) = -f(-x)$ for any $x \in \mathbb{R}$.