Analysis Exam, August 2020

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Let K be a compact metric space and $F : K \times [0,1] \to \mathbb{C}$ a continuous function. Define $f_n, f : K \to \mathbb{C}$ by
$$f_n(x) = F\left(x, \frac{1}{n}\right), \qquad f(x) = F(x, 0).$$
Prove that the sequence f_n converges uniformly to the function f.

2. Suppose that $M \subset [0,1]^n$ is a Borel set with positive Lebesgue measure. Prove that there is some point $x \in \mathbb{R}^n$ such that, for every coordinate vector e_i, the line ℓ_i through x in direction e_i has the property that $\ell_i \cap M$ is a Borel subset of $\mathbb{R}$ and has positive measure.

3. Suppose that $f : [-1,1] \to \mathbb{R}$ is a nonnegative C^∞ function with $f(-1) = f(1) = 0$. Let f^* be the unique nonnegative function which is radially symmetric ($f^*(x) = f^*(y)$ for all $|x| = |y|$), which is nonincreasing ($f^*(x) \geq f^*(y)$ for $|x| \leq |y|$), and such that $f^{-1}((c,\infty))$ has the same Lebesgue measure as $(f^*)^{-1}((c,\infty))$ for all $c \in \mathbb{R}$. You may use the fact that f^* is C^∞ without proof.
   (a) Suppose that $p \geq 1$. How does $\|f\|_{L^p}$ compare to $\|f^*\|_{L^p}$?
   (b) Prove that
$$\int_{-1}^{1} |f'(x)| dx \geq \int_{-1}^{1} |(f^*)'(x)| dx.$$

4. Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Let $f : \mathbb{C}_+ \to \mathbb{C}$ be an analytic function. Assume that for all $z \in \mathbb{C}_+$ such that $|z| = 1$, $f(z) \in \mathbb{R}$. If f has no zeros with $|z| < 1$, prove that it has no zeros with $|z| > 1$.

5. Let $f(z)$ be an entire holomorphic function. Suppose that there are positive real numbers a, b, and k such that $|f(z)| \leq a + b|z|^k$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is a polynomial.

6. Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. If $f : \{z \in \mathbb{C} \mid |z| > 1\} \to \mathbb{C}_+$ is analytic, prove that the limit $\lim_{z\to\infty} f(z)$ is convergent.