# ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2022

**Instructions:**

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam: "On my honor, I have neither given nor received any unauthorized aid on this (assignment, exam, paper, etc.)."

*Date*: May 3, 2022.

(1) (a) Provide one characterization of a group being *solvable*.

(b) Provide one characterization of a group being *nilpotent*.

(c) Which of the following statements is true?

(i) A solvable group need <u>not</u> be nilpotent; or

(ii) A nilpotent group need <u>not</u> be solvable.

Provide an example, with reasoning, to verify your choice.

(2) Consider the polynomials $f_n(x) = x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1 \in \mathbb{Z}[x]$.

(a) Determine if $f_5(x)$ is irreducible.

(b) Determine if $f_6(x)$ is irreducible.

(3) Let $V$ be a finite-dimensional real vector space, and let $B : V \times V \to \mathbb{R}$ be a bilinear form on $V$. The *left radical* of $B$ is the following subspace of $V$:

$$V^{\perp \ell} := \{v \in V \mid B(v, v') = 0 \ \ \forall v' \in V\}.$$

(a) Prove that $B$ is nondegenerate if and only if $V^{\perp \ell} = 0$.

(b) If $B$ is nondegenerate, what is the rank of $B$?

(c) If $B$ is nondegenerate, symmetric, and positive definite, what is the signature of $B$?

(4) Let $\zeta_{13} = e^{2\pi i/13} \in \mathbb{C}$ be a primitive 13th root of unity.

(a) Show that 2 is a primitive root of unity modulo 13.

(b) Find, with proof, a primitive element for the intermediate field extension

$$\mathbb{Q} \subset K \subset \mathbb{Q}(\zeta_{13})$$

such that $[K : \mathbb{Q}] = 2$. Express your element in terms of $\zeta_{13}$.

(5) Let $R$ be a commutative ring with unit, and let $f \colon M \to M'$ be an $R$-module homomorphism.

(a) Prove that $M = 0$ if and only if the localizations $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset R$.

(b) Show that $f$ is injective if and only if the induced maps $f_{\mathfrak{m}} \colon M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ are injective for all maximal ideals $\mathfrak{m} \subset R$.

(6) Let $R$ be a ring, and let $P$ be a left $R$-module.

(a) List three characterizations of $P$ being a *projective* $R$-module.

(b) Prove that one of the characterizations in part (a) implies another in part (a).