ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2022

Instructions:

• You should complete this exam in a single four block of time. Attempt all six problems.
• The use of books, notes, calculators, or other aids is not permitted.
• Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
• Write and sign the Honor Code pledge at the end of your exam.

Date: January 10, 2022.
(1) Show that any group \( G \) of order 32 has center not equal to \( \langle e \rangle \).

(2) Let \( R \) be a commutative ring with 1.
   (a) Suppose that \( P \) is a prime ideal of \( R \). Is \( P[x] \) a prime ideal of \( R[x] \)? Prove or provide a counterexample.
   (b) Suppose that \( M \) is a maximal ideal of \( R \). Is \( M[x] \) a maximal ideal of \( R[x] \)? Prove or provide a counterexample.

(3) Describe all of the \( \mathbb{Z} \)-module homomorphisms from \( \mathbb{Z}/6\mathbb{Z} \) to \( \mathbb{Z}/9\mathbb{Z} \).

(4) Let \( p(x) \) be an irreducible polynomial over \( \mathbb{Q} \).
   (a) Compare the degree of \( p(x) \) and the order of the Galois group \( G \) of \( p(x) \).
   (b) Provide an example to show that we can have \( \deg(p(x)) \neq |G| \) when \( G \) is nonabelian. Include brief reasoning.
   (c) Show that \( \deg(p(x)) = |G| \) when \( G \) abelian.

(5) Let \( R \) be an integral domain, with field of fractions \( F \). Show that \( F = R \) if and only if \( F \) is a finitely generated \( R \)-module.

(6) Let \( V \) be an \( n \)-dimensional complex vector space. Given the characteristic polynomial of a linear operator \( T : V \to V \), determine the characteristic polynomial of the linear operator \( T^\wedge m : \wedge^m V \to \wedge^m V \), for any integer \( m > 1 \).