ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2022

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 10, 2022.

(1) Show that any group G of order 32 has center not equal to $\langle e \rangle$.

(2) Let R be a commutative ring with 1.

(a) Suppose that P is a prime ideal of R. Is $P[x]$ a prime ideal of $R[x]$? Prove or provide a counterexample.

(b) Suppose that M is a maximal ideal of R. Is $M[x]$ a maximal ideal of $R[x]$? Prove or provide a counterexample.

(3) Describe all of the $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/6\mathbb{Z}$ to $\mathbb{Z}/9\mathbb{Z}$.

(4) Let $p(x)$ be an irreducible polynomial over $\mathbb{Q}$.

(a) Compare the degree of $p(x)$ and the order of the Galois group G of $p(x)$.

(b) Provide an example to show that we can have $\deg(p(x)) \neq |G|$ when G is nonabelian. Include brief reasoning.

(c) Show that $\deg(p(x)) = |G|$ when G abelian.

(5) Let R be an integral domain, with field of fractions F. Show that $F = R$ if and only if F is a finitely generated R-module.

(6) Let V be an n-dimensional complex vector space. Given the characteristic polynomial of a linear operator $T : V \to V$, determine the characteristic polynomial of the linear operator

$$T^{\wedge m} : \wedge^m V \to \wedge^m V,$$

for any integer $m > 1$.