ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2020

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 21st, 2020.

(1) Let $G := \mathrm{SL}_2(\mathbb{F}_3)$ be the group of 2×2 matrices of determinant 1 over the field with three elements.

   (a) Determine the order of G.

   (b) Prove that the subgroup $H < G$ generated by

   $$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

   is a 2-Sylow subgroup of G.

   (c) Is the subgroup H normal in G? Justify your answer.

(2) Let M be an $n \times n$ matrix with entries in $\mathbb{C}$, and let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of M. Prove that $\lambda_1^k, \dots, \lambda_m^k$ are eigenvalues of M^k. Can the matrix M^k have an eigenvalue $\lambda \notin \{\lambda_1^k, \dots, \lambda_m^k\}$?

(3) Let $\zeta_{13} = e^{2\pi i/13} \in \mathbb{C}$ be a primitive 13-th root of unity, and let $K = \mathbb{Q}(\zeta_{13})$.

   (a) Determine the order of 2 as an element of the multiplicative group of units of $\mathbb{Z}/13\mathbb{Z}$.

   (b) Determine the lattice of proper subfield extensions for $K/\mathbb{Q}$, i.e., determine all the proper intermediate extensions between $\mathbb{Q}$ and K by giving a primitive element for each extension, as well as any inclusions between these subfields.

(4) Let $\mathfrak{p} \subset \mathbb{Z}[x]$ be a nonzero prime ideal.

   (a) Show that $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.

   (b) Suppose that $\mathfrak{p} \cap \mathbb{Z} = (0)$. Prove that $\mathfrak{p}$ is a principal ideal. [Hint: consider the localization $S^{-1}\mathfrak{p}$, where $S = \mathbb{Z} \setminus \{0\}$.]

   (c) Now suppose that $\mathfrak{p} \cap \mathbb{Z} = (p)$ for a prime number p. Prove that $\mathfrak{p} = (p)$ or $(p, f(x))$, where $f(x)$ is irreducible.

(5) A division ring is a nonzero ring A with unit 1_A (not necessarily commutative) such that every nonzero element has a (necessarily) two-sided multiplicative inverse, i.e., for $0 \neq a \in A$, there exists $b \in A$ such that $a \cdot b = b \cdot a = 1_A$.

   Let R be a commutative ring. Recall that an R-module M is said to be simple if it is nonzero and its only R-submodules are 0 and itself. Show that the endomorphism ring, $\mathrm{End}_R(M)$, of a simple R-module is a division ring.

(6) Find fields K_1 and K_2 such that

   $$\mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{3}) \simeq K_1 \times K_2.$$

   Justify your answer.