ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2023

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 12, 2023.

(1) Prove that there is no simple group of order 280.

(2) Let R be an integral domain, not a field. We say that R is an *Euclidean Domain* if there exists a function:

$$N : R \setminus \{0\} \to \mathbb{N} \cup \{0\}$$

satisfying the following conditions:

(i) $N(a) \leq N(ab)$ where $a, b \in R$ and $ab \neq 0$;

(ii) for all $a, b \in R$ with $a \neq 0$, there exists $q, r \in R$ with $b = qa + r$, where either $r = 0$ or $N(r) < N(a)$.

Complete the problems below.

(a) Prove that a Euclidean Domain is a Principal Ideal Domain.

(b) Suppose that R is a Euclidean Domain. Prove that R contains an element u which is not a unit of R and satisfies the following property:

(*) For every $x \in R$, either $u|x$ or there is a unit $z \in R$ such that $u|(x + z)$.

(c) Assuming that $\mathbb{Z}[i]$ is a Euclidean domain under $N(a + bi) = a^2 + b^2$, exhibit such an element $u \in \mathbb{Z}[i]$ as in part (b).

(3) Let G and G' be finite abelian groups such that the greatest common divisor of $|G|$ and $|G'|$ is equal to 1. Simplify the tensor product of $\mathbb{Z}$-modules: $G \otimes_{\mathbb{Z}} G'$.

(4) Consider the following Galois groups.

(i) The Galois group of the splitting field of $x^4 - 1$ over $\mathbb{Q}$.

(ii) The Galois group of the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

(iii) The Galois group of the splitting field of $x^4 - 3$ over $\mathbb{Q}$.

(iv) The Galois group of the splitting field of $x^4 - 4$ over $\mathbb{Q}$.

Which pairs of the groups above are isomorphic? Provide justification.

(5) Let R be a commutative domain with a subring R', and suppose that R is integral over R'.

(a) Show that if I is an ideal of R, then R/I is integral over $R'/(R' \cap I)$.

(b) Let S' be a multiplicatively closed subset of R'. Show that the ring of fractions $S'^{-1}R$ is integral over $S'^{-1}R'$.

(6) Prove the statements below or provide a counterexample. Let R be a commutative ring.

(a) If there exists an ideal I of R such that R/I is Noetherian, then R is Noetherian.

(b) If the polynomial extension $R[x]$ of R is Noetherian, then R is Noetherian.