ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2022

Instructions:

- You should complete this exam in a single **four hour** block of time. Attempt all **six** problems.

- The use of books, notes, calculators, or other aids is **not** permitted.

- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.

- Write and sign the Honor Code pledge at the end of your exam: "On my honor, I have neither given nor received any unauthorized aid on this (assignment, exam, paper, etc.)."

For questions, email:

Tony before 2pm and after 3:15pm - av15@rice.edu

Brandon between 2pm and 3pm - bwlevin@rice.edu

When you've completed the exam, scan and email it to Chris at cjl12@rice.edu and drop off the hard copy in his office.

Date: August 16, 2022.

(1) Let $p < q$ be distinct prime numbers. Prove that any group of order pq is solvable.

(2) Let $I = \langle x^2 + xy^2, x^2 - y^3, y^3 - y^2 \rangle$ be an ideal in the polynomial ring $\mathbb{Q}[x, y]$. Fix the lexicographic ordering $x > y$ in this ring.

(a) Define what is meant by a reduced Gröbner basis for I.

(b) Show that $\{x^2 - y^2, y^3 - y^2, xy^2 + y^2\}$ is a reduced Gröbner basis for I.

(3) Let p be a prime number. Consider the following matrices in $G := \mathrm{GL}_3(\mathbb{F}_p)$

$$A := \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix} \qquad \text{and} \qquad B := \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Suppose that $p = 5$. Are A and B conjugate in G? Justify.

(b) Suppose that $p = 7$. Are A and B conjugate in G? Justify.

(4) Let $\alpha = 2\sin(2\pi/5)$, and let $K = \mathbb{Q}(\alpha)$.

(a) Show that the minimal polynomial of α over $\mathbb{Q}$ is $f(x) := x^4 - 5x^2 + 5$.

(b) Show that $\sqrt{5} \in K$.

(c) Is the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois? Justify. [Hint: $\sqrt{(5 - \sqrt{5})(5 + \sqrt{5})} = 2\sqrt{5}$.]

(5) Let R be a commutative ring with unit, and let N be a finitely generated R-module. Let $\mathcal{M} = (M_i, \mu_{ij})$ be a directed system of R-modules.

(a) Explain how to construct a natural map

$$\varinjlim \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R\left(N, \varinjlim M_i\right).$$

(b) Prove that the map is an isomorphism if N is free. [Hint: start with the case $N = R$.]

(6) Let $A \subseteq B$ be an extension of commutative rings with unit. Suppose that B is integral over A. Prove that B is a field if and only if A is a field.