RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2017

This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map and $n > m$. Prove that f cannot be one-to-one.

2. Let $F(n)$ be the free group of rank n. For each integer $n \geq 2$, prove that $F(2)$ contains a finite index normal subgroup isomorphic to $F(n)$.

3. Let $f : H \to \mathbb{R}^3$ be a topological embedding where H is a solid handlebody of genus 2 (i.e. a thickened θ graph) as in Figure 1. Let $X = \text{int}(f(H))$ be the image of f in $\mathbb{R}^3$. Compute $H_p(\mathbb{R}^3 - X)$ for all p.

FIGURE 1. H, a solid handlebody of genus 2

4. Let $T = S^1 \times S^1$ and let $f : T \to T$ be defined by
$$f(x, y) = (2x + y, x + y).$$
Here we are viewing S^1 as $\mathbb{R}/\mathbb{Z}$. Let $X = (T \times [0, 1])/\sim$ be the 3-manifold obtain by identifying $(x, y) \times \{0\}$ with $f(x, y) \times \{1\}$. Compute $\pi_1(X)$.

5. Let M be a closed, connected, orientable 4-dimensional manifold with $\pi_1(M) \cong \mathbb{Z}_5 * \mathbb{Z}_5$ and $\chi(M) = 5$.
   (a) Compute $H_p(M; \mathbb{Z})$ for all p.
   (b) Prove that M is not homotopy equivalent to a CW complex with no 3-cells.

6. Let $n \geq 1$. Prove that there is no orientation-reversing (that is degree -1) map $f : \mathbb{C}P(2n) \to \mathbb{C}P(2n)$.