RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2016

This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let $r : X \to A$ be a retract and X be a contractible space. Prove that A is contractible.

2. Let M be a closed, smooth, simply-connected n-dimensional manifold and let $T = S^1 \times \cdots \times S^1$ be the n-dimensional torus. Prove that there does not exist a smooth immersion from M to T.

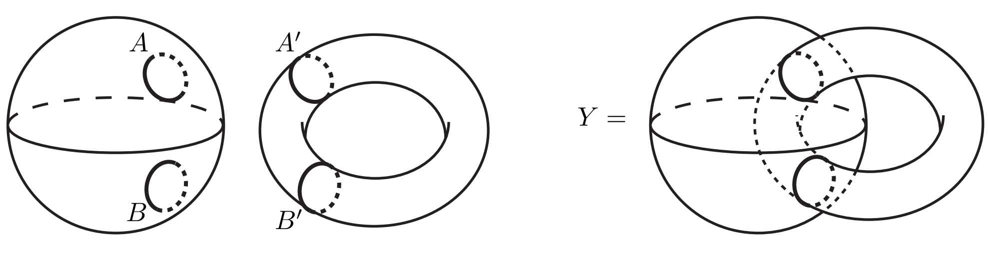
3. Let $W = S^1 \vee S^1$ be the wedge of 2 circles. Describe four distinct connected 3-fold covering spaces of W including at least one irregular cover. In each case, give the group of covering transformations, say whether or not the covering is regular and give the corresponding subgroup of $\pi_1(W)$.

4. Let H be a solid handlebody of genus 2. Recall that H can be obtained as follows. Let S be the surface (with boundary) pictured in Figure 1, then $H = S \times I$. H can also be viewed as a 3-dimensional manifold obtained by thickening up a wedge of circles (its boundary is a genus 2 surface).

   Let $f : H \to \mathbb{R}^3$ be a topological embedding. Let $X = \text{int}(f(H))$ be the image of f in $\mathbb{R}^3$. Compute $H_p(\mathbb{R}^3 - X)$ for all p.

FIGURE 1. S

5. Suppose Y is a topological space which is obtained from the union of a 2-sphere S^2 and a torus T by identifying the circle A to the circle A' and the circle B to the circle B' as shown below. Thus $S^2 \cap T \cong S^1 \sqcup S^1$.



   a) Calculate $H_p(Y; \mathbb{Z})$ for all p.
   b) Sketch or describe "geometric" representatives of the generators of $H_1(Y; \mathbb{Z})$ and $H_2(Y; \mathbb{Z})$.
   c) Calculate $\pi_1(X)$.
   d) Sketch or describe a connected 2-fold covering space of Y and the covering map.

6. Let M be a closed, connected, oriented 4-dimensional manifold with $b_2(M) \geq 1$. Prove that any continuous map $f : S^4 \to M$ has degree 0.