January 2010 - TOPOLOGY QUALIFYING EXAM - RICE UNIVERSITY

This is a 3 hour, closed book, closed notes exam.

1. Determine exactly which cyclic groups can act freely and properly discontinuously on the closed orientable surface, Σ_8, of genus 8. Which of these can act on Σ_8 with non-orientable quotient?

2. a) Fix $n \geq 1$. Let S_i, $i = 1, \ldots, k$ be a copy of the n-sphere, S^n, with north pole n_i and south pole s_i. Let X be the quotient space obtained from the disjoint union $\coprod_{i=1}^{k} S_i$ by identifying n_i with s_{i+1} for $i = 1, ..., k-1$ and identifying n_k with s_1. Determine (with full argument) $\pi_1(X)$.

   b) Compute $H_*(X;\mathbb{Z})$ by applying a Mayer-Vietoris sequence directly to the space X.

3. Let X be a CW complex obtained by starting with a circle, Σ, and then adjoining (to Σ) two 2-cells e_i, $i = 1, 2$, by maps $f_i : \partial(e_i) \to \Sigma$ of degree 6 and 8 respectively.

   a) Calculate $H_*(X;\mathbb{Z})$.

   b) Calculate $H_*(X, \Sigma;\mathbb{Z})$.

   c) Calculate a cellular cochain complex with $\mathbb{Z}_3$-coefficients for X and use it to calculate $H^*(X;\mathbb{Z}_3)$.

   d) Calculate $\pi_2(X)$. Hint: Find a cell structure for the universal cover of X.

4. Give an example for each of the following **or** state "such an example does not exist because...(give reason)." No justification needed except as noted.

   a) Two spaces with isomorphic π_1 but non-isomorphic integral homology groups; Which homology group distinguishes them?

   b) Two spaces with isomorphic integral homology groups but non-isomorphic π_1; Give π_1 of the spaces.

   c) Two spaces with isomorphic integral homology groups but non-isomorphic cohomology groups. Which cohomology group distinguishes them?

   d) Two spaces with isomorphic π_1 and isomorphic integral homology groups which are NOT homotopy equivalent; State why they are not homotopy equivalent in one or two sentences).

   e) Two spaces that are homotopy equivalent but not homeomorphic.

5. If n is even, prove that there is no orientation-reversing (that is, degree -1) homotopy equivalence $f : \mathbb{C}P(n) \to \mathbb{C}P(n)$.