RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2016

This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let M be a smooth n-dimensional manifold.
   (a) Prove that the tangent bundle TM is a smooth $2n$-dimensional manifold.
   (b) Prove that TM is orientable (whether or not M was orientable).

2. Suppose X is a path-connected topological space with a universal covering space $\tilde{X}$ that is compact. Prove that $\pi_1(X)$ is finite.

3. Let X be the space obtained from a circle by attaching two 2-cells, one using a map of degree 6 and the other using a map of degree 8.
   (a) Compute $\pi_1(X)$.
   (b) Calculate $H^p(X;\mathbb{C})$ and $H_p(X;\mathbb{Z}_2)$, for all p, without using a universal coefficient theorem. Your answer should include some definition of cohomology and homology with coefficients.
   (c) Prove that X is not homotopy equivalent to any compact, connected n-dimensional manifold without boundary.

4. Let F be a free group of rank n and let S be a subgroup of F of finite index d.
   (a) Prove that S is a free group.
   (b) Calculate, with proof, the rank (as a free group) of S in terms of n and d.

5. Let X be a compact, connected, orientable 4-dimensional manifold without boundary such that $\pi_1(X) \cong \mathbb{Z}_{15}$ and $H_2(X;\mathbb{Q}) \cong \mathbb{Q}^2$. Let E be a connected 3-fold covering space of X.
   (a) Calculate $\pi_1(E)$.
   (b) Calculate $H_p(X;\mathbb{Z})$ for each p.
   (c) Calculate $\chi(X)$.
   (d) Calculate $H_p(E;\mathbb{Z})$ for each p.
   (e) Prove that E admits no CW decomposition without 3-cells.

6. Let X and Y be compact, connected, oriented n-dimensional manifolds without boundary and let $f : X \to Y$ be a continuous map. Suppose $\beta_p(X) < \beta_p(Y)$ for some $p > 0$.
   (a) Prove that $f^* : H^p(Y;\mathbb{Q}) \to H^p(X;\mathbb{Q})$ has a non-trivial kernel.
   (b) Show that f is a degree zero map.