Rice University Topology Qualifying Exam

May, 2015

This is a 3 hour, closed book, closed notes exam. For maximum credit include justification for all steps. Sign the Rice honor pledge at the end of your exam.

1. Let $f : S^n \to S^n$ be a smooth map with no fixed points. Prove that the degree of f is $(-1)^{n+1}$.

2. Let L_1 and L_2 be disjoint straight lines in $\mathbb{R}^3$. Calculate $\pi_1(\mathbb{R}^3 \setminus (L_1 \cup L_2))$.

3. Let $W = S^1 \vee S^1$. Construct three connected 3-fold covers of W that are distinct up to covering space equivalence, including at least 1 irregular cover. For each of these three covers, describe the covering map, say whether or not the cover is regular, and give the corresponding subgroup of $\pi_1(W)$.

4. For $n \geq 2$, prove that for any continuous map $f : \mathbb{CP}^n \to S^2$, the induced map $f_* : H_2(\mathbb{CP}^n; \mathbb{Z}) \to H_2(S^2; \mathbb{Z})$ is the zero map.

5. Give an example (a CW complex) for each of the following or state that such an example does not exist. Give a brief justification in all cases.

   (a) Two spaces with isomorphic π_1 but non-isomorphic integral homology groups.

   (b) Two spaces with isomorphic integral homology groups but non-isomorphic π_1 (give π_1 of the spaces).

   (c) Two spaces with isomorphic integral homology groups but non-isomorphic cohomology groups.

   (d) Two spaces that are homotopy equivalent but not homeomorphic.

   (e) Two spaces with isomorphic π_1 and isomorphic integral homology groups that are NOT homotopy equivalent.

6. Let M be a compact contractible n-manifold with boundary. Prove that ∂M is a homology $(n-1)$-sphere, i.e. that $H_i(\partial M; \mathbb{Z}) \cong H_i(S^{n-1}; \mathbb{Z})$ for all i.