May 2014 - TOPOLOGY QUALIFYING EXAM - RICE UNIVERSITY

This is a 4 hour, closed book, closed notes exam. For maximum credit include justification for all steps. SIGN the RICE HONOR PLEDGE at the end of your exam.

1. (This problem is worth roughly half of the value of the other problems) Suppose the following is an exact sequence of **abelian** groups:

$$0 \to \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}_9 \xrightarrow{h} \mathbb{Z}_3 \xrightarrow{\phi} \mathbb{Z}_{25}$$

Deduce, with proof, all possible values for the isomorphism type of A.

2. Let X be the space obtained from a solid octagon by identifying sides as shown in Figure 1 below.

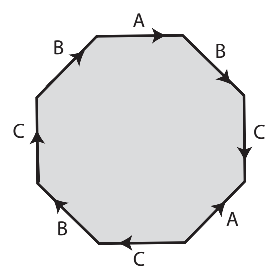

Figure 1

(a) Give a CW-structure for X (be careful with the vertices) and describe the cellular chain complex, including the precise maps.
(b) Give a presentation for $\pi_1(X)$ (be careful with the vertices) .
(c) Calculate $H_n(X; \mathbb{Z}_3)$ and $H^n(X; \mathbb{C})$ for all $n \geq 0$.

3. Let Σ be the closed, orientable surface of genus $g \geq 1$. Prove that any continuous map $f : \mathbb{R}P(2) \times \mathbb{R}P(2) \to \Sigma$ is null-homotopic. Hint: Use covering spaces.

4. (This problem is worth roughly half of the value of the other problems) Prove that any continuous map $f : S^4 \to \mathbb{C}P(2)$ induces the zero map on $H_4(-; \mathbb{Z})$.

5. Let X be a connected, orientable, compact 4-dimensional manifold without boundary such that $\pi_1(X) \cong \mathbb{Z}_{15}$ and $\chi(X) = 3$.
   a) Calculate, with explanation, $H_i(X; \mathbb{Z})$ for all i.
   b) Calculate, with explanation, the integral cohomology ring of X.
   c) Prove that every CW structure for X has some 3-cells.

6. a) Let $\vec{v} = (a, b)$ be a fixed non-zero vector in $\mathbb{R}^2$. Define an equivalence relation on $\mathbb{R}^2$ by, for each $(x, y) \in \mathbb{R}^2$ and integer n, $(x, y) \sim (x, y) + n\vec{v}$. Give the set of equivalence classes the quotient topology and call the resulting topological space B. Is the quotient map $\pi : X \to B$ a covering map? Why or why not?
   b) Calculate $\pi_1(B)$ and $\pi_2(B)$ with explanation.