RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2013

This is a 3-hour closed book, closed notes exam. JUSTIFY all of your work, as time allows.

1. Suppose X is a contractible space.

   (a) If $A \subset X$ is a retract of X, prove that A is contractible.

   (b) Give an example of a contractible 2-dimensional polyhedron X and a connected subspace B of X such that B is not contractible.

2.

   (a) Let $W = S^1 \vee S^1$ be the wedge of 2 circles. Describe two distinct connected 3-fold covering spaces (don't forget the covering map) of W including one irregular cover. In each case, give the group of covering transformations, say whether or not the covering is regular and give the corresponding subgroup of $\pi_1(W)$.

   (b) Perhaps using part (a), describe a connected irregular 3-fold cover of $S^1 \vee \mathbb{R}P(2)$.

3. Let X be a 2-dimensional CW complex constructed as follows: Start with a circle, Σ, and then adjoin (to Σ) two 2-cells e_i, $i = 1, 2$, by maps $f_i : \partial(e_i) \to \Sigma$ of degree 6 and 8 respectively.

   (a) Calculate $H_*(X; \mathbb{Z})$.

   (b) Calculate $H_*(X, \Sigma; \mathbb{Z})$.

   (c) Calculate a cellular cochain complex with $\mathbb{Z}_3$-coefficients for X and use it to calculate $H^*(X; \mathbb{Z}_3)$.

   (d) Calculate $\pi_1(X)$. Calculate $\pi_2(X)$ (Hint: Find a cell structure for the universal cover of X).

4. Let X be a compact, connected, orientable, 4-dimensional manifold without boundary such that $\pi_1(X) \cong \mathbb{Z}_{35}$ and $\beta_2(X) = 4$.

   (a) Calculate $H_i(X; \mathbb{Z})$ for all i.

   (b) Suppose $\widetilde{X}$ is a connected regular 7-fold (7-sheeted) covering space of X. Calculate $\pi_1(\widetilde{X})$ and $H_i(\widetilde{X}; \mathbb{Z})$ for all i.

5. Let $f : S^2 \vee S^4 \to X$ be a continuous map where X is a compact, connected, orientable 4-dimensional manifold without boundary with $\beta_2(X) \neq 0$. Prove that the induced map f_* on $H_4(-; \mathbb{Z})$ is the zero map.