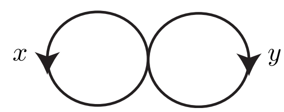
May 2011 - TOPOLOGY QUALIFYING EXAM - RICE UNIVERSITY

This is a 3 hour, closed book, closed notes exam. Please include thorough justifications, as much as time allows. Sign the honor pledge at the conclusion of the exam.

1. Let $X = S^1 \vee S^1$ be a wedge of circles labelled x and y (as indicated below). Let Y be the space obtained by attaching two 2-cells to X along the curves $x^2y^{-1}xy^2$ and yx^5yx, respectively.



   (a) Compute $H_p(Y; \mathbb{Z}_3)$ for all p.
   (b) Compute $H^p(Y; \mathbb{Z})$ for all p without using a Universal Coefficient Theorem.

2. Let X be a closed, connected, orientable 4-dimensional manifold with $\pi_1(X) \cong \mathbb{Z}_{15}$ and $\chi(X) = 5$. Let $\widetilde{X}$ be a connected 3-fold covering space of X. What is $\pi_1(\widetilde{X})$? What is $\chi(\widetilde{X})$? Calculate $H_i(\widetilde{X})$ for each i.

3. Let $X = S^1 \times S^1 - D$ where D is an open 2-dimensional disk embedded in $S^1 \times S^1$.
   (a) What is $\pi_1(X)$?
   (b) Let $\tilde{X}$ be the connected covering space of X corresponding to $[\pi_1(X), \pi_1(X)]$, the commutator subgroup of $\pi_1(X)$. Is $\tilde{X}$ a regular cover? What is the group of deck transformations (covering transformations) of $\tilde{X} \xrightarrow{p} X$?
   (c) Draw a picture of $\tilde{X}$ and describe how the deck transformations act (Hint: it might be helpful to consider covering spaces of spaces that are closely related to X).
   (d) Calculate $\pi_n(X)$ for each $n > 1$, with explanation.

4. Let $f : S^2 \times S^6 \to \mathbb{C}P^4$ be a continuous map.
   (a) Show that f induces the zero map on $H^6(-;\mathbb{Z})$ and $H^8(-;\mathbb{Z})$.
   (b) Use (a) to show that f induces the zero map on $H_6(-;\mathbb{Z})$ and $H_8(-;\mathbb{Z})$.

5. (This problem is worth roughly half of the value of the other problems) Suppose the following is an exact sequence of **abelian** groups:
$$0 \to \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}_9 \xrightarrow{h} \mathbb{Z}_3 \xrightarrow{\phi} \mathbb{Z}_5$$
   Deduce, with proof, all possible values for the isomorphism type of A.

6. Suppose (X, A) is a pair of topological spaces. State, and sketch the proof of, the theorem asserting the existence of a long exact sequence in homology for this pair. The proof sketch is viewed as the major content of this problem. Define the boundary homomorphism (the one that decreases degree).