August 2013 - TOPOLOGY QUALIFYING EXAM - RICE UNIVERSITY

This is a 3 hour, closed book, closed notes exam. Please include thorough justifications, as much as time allows. Sign the honor pledge at the conclusion of the exam.

(1) Let X be a 2-dimensional CW complex constructed as follows: Start with a circle, Σ, and then adjoin (to Σ) two 2-cells e_i, $i = 1, 2$, by maps $f_i : \partial(e_i) \to \Sigma$ of degree 10 and 25 respectively.

(a) Calculate $H_*(X; \mathbb{Z})$.

(b) Calculate $H_*(X, \Sigma; \mathbb{Z})$.

(c) Calculate a cellular cochain complex with $\mathbb{Z}_2$-coefficients for X and use it to calculate $H^*(X; \mathbb{Z}_2)$.

(d) Calculate $\pi_1(X)$. Calculate $\pi_2(X)$ (Hint: Find a cell structure for the universal cover of X).

(2) Suppose X is a 2-dimensional disk with two open sub-disks removed. Discuss all possible connected **2-fold** covering spaces of X. Your discussion should include: how many are there (up to covering space isomorphism); how do these relate to $\pi_1(X)$?, describe them, what are their groups of covering transformations, are they regular (normal) covering spaces?

(3) Let X be a closed, connected, orientable 4-dimensional manifold with $\pi_1(X) \cong \mathbb{Z}_{15}$ and $H_2(X; \mathbb{Q}) \cong \mathbb{Q}^3$. Let E be a connected 3-fold covering space of X.
(a) What is $\pi_1(E)$?
(b) Calculate $H_i(X; \mathbb{Z})$ for each i. What is $\chi(X)$?
(c) Calculate $H_i(E; \mathbb{Z})$ for each i.
(d) Prove that E admits no CW decomposition without 3-cells.

(4) Let $f : \mathbb{C}P(n) \to \mathbb{C}P(n)$ be a continuous map. Prove that the degree of f is m^n for some integer m.

(5) Determine exactly which cyclic groups can act (freely and) properly discontinuously on the closed orientable surface, Σ_4, of genus 4. Which of these can act on Σ_4 with orientable quotient?

(6) Suppose (X, A) is a pair of topological spaces. State, and sketch the proof of, the theorem asserting the existence of a long exact sequence in homology for this pair. The proof sketch is viewed as the major content of this problem. Define the boundary homomorphism (the one that decreases degree).