RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2008

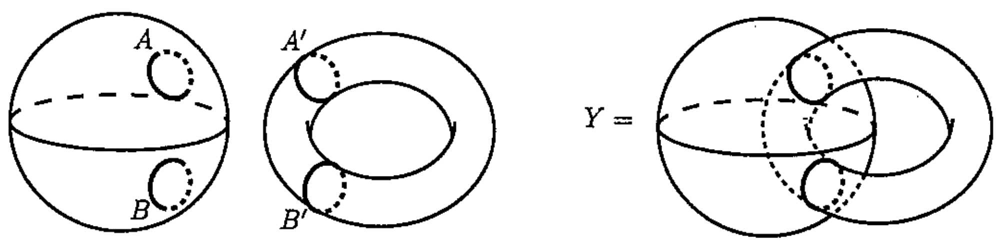
This is a 3-hour closed book, closed notes exam. Please show all of your work.

1. A continuous map $f : X \to Y$ is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. Prove that the image of a proper map $f : \mathbb{R}^n \to \mathbb{R}$ is closed.

2. Let D_p be the p-fold dunce cap obtained by attaching a 2-cell to the circle by the attaching map $f : S^1 \to S^1$ defined by $f(z) = z^p$ (we are considering $z \in S^1$, the boundary of the unit disk in $\mathbb{C}^2$).
 a) Describe a cellular chain complex for D_p.
 b) Suppose p is a prime integer. Let $\mathbb{Z}[\frac{1}{p}] := \{\frac{m}{p^k} \mid m \in \mathbb{Z} \text{ and } k \in \mathbb{Z}_+ \cup 0\}$ be the subring of $\mathbb{Q}$. Note that $\mathbb{Z}[\frac{1}{p}]$ is the smallest subring of $\mathbb{Q}$ in which p has a multiplicative inverse. Using part (a), calculate (with proof) $H_i(D_p; \mathbb{Z}[\frac{1}{p}])$ for all $i \geq 0$. *Do not use a Universal Coefficient Theorem.*

3. Let $Y = D^2 - \{0\}$ and $X = Y \times \mathbb{R}P^2$, where $\mathbb{R}P^2$ is the real projective plane. Briefly discuss all of the covering spaces of X. How "many" are there? Describe them. What are their groups of covering transformations? Which ones are regular (normal)?

4. Suppose Y is a topological space which is obtained from the union of a 2-sphere S^2 and a torus T by identifying the circle A to the circle A' and the circle B to the circle B' as shown below. Thus $S^2 \cap T \cong S^1 \sqcup S^1$.



 a) Calculate $H_i(Y; \mathbb{Z})$ for all i.
 b) Sketch or describe "geometric" representatives of the generators of $H_1(Y; \mathbb{Z})$ and $H_2(Y; \mathbb{Z})$.
 c) Calculate $\pi_1(X)$.
 d) Sketch or describe a connected 2-fold covering space of Y and the covering map.

5. Let X be a closed, oriented 4-manifold with $\pi_1(X) \cong \mathbb{Z}/15\mathbb{Z}$ and $\chi(X) = 5$. Let $\widetilde{X}$ be a connected 3-fold covering space of X. Calculate $H_i(\widetilde{X}; \mathbb{Z})$ for all i.

6. Let X be a closed (compact and boundaryless), oriented 4-manifold with $\beta_2(X) \neq 0$. Prove that any continuous map $f : S^4 \to X$ has degree equal to 0.