Analysis Preliminary Exam, January 2015, 3 hours

1. Justifying all your steps, evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{dx}{(1+\frac{x}{n})^n x^{1/n}}$$

where dx denotes integration with respect to Lebesgue measure.

2. Suppose u is a real-valued function such that $f = u + \mathbf{i}u^2$ is holomorphic. Prove that u is necessarily a constant.

3. Suppose f_k and f are integrable functions on $[0,1]$ and, as $k \to \infty$, $f_k \to f$ a.e. and $\int_0^1 f_k\, dx \to \int_0^1 f\, dx$.

(a) Give an example of such f_k, f for which $\int_0^1 |f_k - f|\, dx \not\to 0$.

(b) Show that if, in addition, each $f_k \geq 0$, then $\int_0^1 |f_k - f|\, dx \to 0$.
[Hint: Consider $h_k = f + f_k - |f - f_k|$.]

4. Suppose that g is a holomorphic function on the punctured unit disk $\{z \in \mathbf{C} : 0 < |z| < 1\}$ satisfying the estimate $|g'(z)| \leq |z|^{-3/2}$.
Show that 0 is then a removable singularity.

5. Suppose $(X, \mathcal{M}, \mu)$ is a measure space , and $E_k \in \mathcal{M}$ is a sequence of measurable subsets with $\mu(E_k) > 10^{-3}$.

(a) Show that the set F of all points of X which lie in infinitely many of the E_k is a measurable set (i.e. belongs to $\mathcal{M}$).

(b) Show that if $\mu(X) < \infty$, then $\mu(F) \geq 10^{-3}$.

(c) Give an example to show that it is possible for $\mu(F) < 10^{-3}$ if one has $\mu(X) = \infty$.

6. Let f_n, g_n be entire functions on $\mathbf{C}$. Assume $f_n(z)g_n(z) = z$ and that f_n converges to f uniformly on compact subsets of $\mathbf{C}$ with f not being identically zero. Show that g_n converges uniformly on compact subsets of $\mathbf{C}$.