RICE

ANALYSIS QUALIFYING EXAM
1-11-11
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1. Let a, b, c be nonzero real numbers. Calculate

$$\int_{-\infty}^{\infty} \frac{dx}{(x-ia)(x-ib)(x-ic)}$$

(as usual, $i=\sqrt{-1}$).

2. a. Let $0<a<1$, and calculate $\displaystyle\int_0^\infty \frac{x^{a-1}}{x+1}\,dx$.

b. Use part a to calculate $\displaystyle\int_0^\infty \frac{x^{a-1}\log x}{x+1}\,dx$.

c. When $a=\frac{1}{2}$ part b gives the result 0. Verify directly by a change of variable that $\displaystyle\int_0^\infty \frac{x^{-1/2}\log x}{x+1}\,dx = 0$.

3. Let $E \subset [0,1]$ and consider the indicator function

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Prove that χ_E is Riemann integrable and $\int_0^1 \chi_E(x)\,dx = 0$ if and only if ________________________.

(You should insert the missing condition.)

4. Let f be a bounded real valued function on $[a,b]$. Prove that f is Lebesgue measurable if and only if for every $\varepsilon > 0$ there exist Lebesgue measurable simple functions g and h such that $g \le f \le h$ and $\int_a^b (h-g)\,dx < \varepsilon$.

5. Let $f_1, f_2, f_3, f_4, \cdots$ be a sequence of Lebesgue measurable functions defined on $[0,1]$ such that

$$|f_n(x)| \le 1 \text{ for all } n \ge 1 \text{ and all } 0 \le x \le 1,$$

and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ exists for each } 0 \le x \le 1.$$

Prove that

$$\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{\sqrt{|x-\frac{1}{n}|}}\,dx = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx.$$

6. Let f be the Riemann map from the unit disk $|z|^2 = x^2+y^2 < 1$ onto the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$,

normalized by the two conditions

$$f(0)=0, \quad f'(0) > 0.$$

Then of course

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \text{ for } |z|<1,$$

and $a_1 > 0$.

Prove that

$$\begin{cases} a_n = 0 \text{ for all even } n, \\ a_n \in \mathbb{R} \text{ for all odd } n. \end{cases}$$