Analysis Exam, January 2008

1. (a) Construct a dense open subset U of $\mathbf{R}$ with Lebesgue measure $\lambda(\mathrm{U}) < 1$.

(b) Does there exist a measurable set E of positive measure with the property that

$$\lambda(E \cap [a,b]) \leq \frac{1}{2}(b-a) \quad \textit{for all} \quad -\infty < a < b < \infty \,?$$

If so, give an example. If not, give a reason why not.

2. Compute the Principal Value integral

$$P.V. \int_{-\infty}^{+\infty} \frac{1}{x^3-1}dx \;=\; \lim_{\varepsilon \downarrow 0}\Big[\int_{-\infty}^{1-\varepsilon} \frac{1}{x^3-1}dx + \int_{1+\varepsilon}^{+\infty} \frac{1}{x^3-1}dx\Big] \,.$$

3. Suppose

$$F(z) \;=\; a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n} + \cdots$$

where

$$\sum_{n=2}^{\infty} n|a_n| \leq |a_1| \,.$$

(a) Prove that F is holomorphic on the exterior region $\Omega = \{z \in \mathbf{C} \;:\; |z| > 1\}$.

(b) Prove that $\lim_{z\to\infty} F(z) = a_0$.

(c) Prove that F is one-to-one on Ω.

4. Suppose $f_1 \leq f_2 \leq f_3 \leq f_4 \leq \cdots$ are real-valued differentiable functions on $\mathbf{R}$ which satisfy

$$f_n(0) = 0 \;\; \textit{and} \;\; -1 \leq f_n'(t) \leq +1 \quad \textit{for all } n = 1, 2, \cdots \;\; \textit{and } t \in \mathbf{R} \,.$$

(a) Show that $f(t) = \lim_{n\to\infty} f_n(t)$ exists and is finite for all $t \in \mathbf{R}$.

(b) Is f necessarily continuous? If so, prove it. If not, find a counterexample.

(c) Is f necessarily differentiable? If so, prove it. If not, find a counterexample.

5. (a) Is it true of the functions f_n from problem 4 that

$$\int_a^b f(t)\,dt \;=\; \lim_{n\to\infty} \int_a^b f_n(t)\,dt \;\; \textit{for all} \quad -\infty < a < b < \infty \,.?$$

If so, then prove it. If not, find a counterexample.

(b) Prove that, for any finite sequence $-\infty < a_0 < a_1 < \cdots < a_n < \infty$, one has

$$\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| \;\leq\; \liminf_{n\to\infty} \int_{-\infty}^{\infty} |f_n'(t)|\,dt \,.$$

(c) Give an example of such a sequence where

$$\sup_{-\infty<a_0<a_1<\cdots<a_n<\infty} \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| \;<\; \liminf_{n\to\infty} \int_{-\infty}^{\infty} |f_n'(t)|\,dt \,.$$

6. Is the following function

$$f(z) = \sum_{n=1}^{\infty} (\frac{1}{2})^n \cos(3^n z\pi)$$

holomorphic on the entire complex plane? If so, explain. If not, describe the set of points where this series fails to converge.