ANALYSIS QUALIFYING EXAM May 13, 2010

1. Given that
$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi},$$
calculate
$$\int_{-\infty}^{\infty} e^{-t^2+tw}\,dt$$
for any complex number w.

2. Let f be a <u>Möbius</u> (also called <u>linear</u> <u>fractional</u>) transformation
$$f(z) = \frac{az+b}{cz+d},$$
where a, b, c, d are complex numbers with $ad - bc \neq 0$. It is known that in general if K is a circle or straight line in the complex plane $\mathbb{C}$, then $f(K)$ is also a circle or a straight line.

Now suppose that there exists a circle K with center z_0 such that $f(K)$ is a circle with center $f(z_0)$.

What can you conclude about f?

3. Let $D = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$, and suppose that f is a holomorphic function defined on D. Suppose also that in the sense of integration on $\mathbb{R}^2$

$$\iint_D |f(x+iy)|\, dx\, dy < \infty.$$

What can you conclude about the nature of the singularity of f at $z=0$?

(Is it removable? If so, explain. If not, what is the nature of this singularity?)

4. Prove the "Riemann–Lebesgue lemma" in detail: namely, if f is Lebesgue integrable on $\mathbb{R}$, then

$$\lim_{x \to \infty,\, x \in \mathbb{R}} \int_{-\infty}^{\infty} f(t)\, e^{itx}\, dt = 0.$$

5. The Weierstrass approximation theorem states that every continuous real-valued function on a compact interval $[a,b] \subset \mathbb{R}$ can be uniformly approximated by polynomials.

Given that, suppose f is a C^1 real-valued function on $[a,b]$. That is, f is differentiable on $[a,b]$ and its derivative f' is continuous on $[a,b]$. Let $\varepsilon > 0$. Then prove that there exists a polynomial $P = P(x)$ such that

$$\sup_{a \le x \le b} |f(x) - P(x)| + \sup_{a \le x \le b} |f'(x) - P'(x)| < \varepsilon.$$