Analysis Exam, August 2013

1. Suppose that D and $\bar{D}$ are the open and closed unit disks in $\mathbf{C}$, that $f : \bar{D} \times \bar{D} \to \mathbf{C}$ is continuous, and that, for every fixed point $a \in D$, the two functions

$$z \in D \mapsto f(z,a) \in \mathbf{C} \quad \text{and} \quad w \in D \mapsto f(a,w) \in \mathbf{C}$$

are holomorphic.

(a) Find a formula for $f(0,0)$ in terms of the values $f(z,w)$ where both $|z| = 1, |w| = 1$.

(b) Is it always true that f vanishes identically whenever there is a sequence (z_i, w_i) approaching $(0,0)$ with $f(z_i, w_i) = 0$? Prove if true or find a counterexample if false.

2. Let I be the open unit interval $\{t \in \mathbf{R} \,:\, 0 < t < 1\}$.

(a) Give an example of a function $f : I \to \mathbf{R}$ that is *continuous* but not *uniformly continuous.*

(b) Give an example of a sequence of uniformly continuous functions $g_n : I \to \mathbf{R}$ that is not an *equi-continuous* sequence.

(c) Give an example of a uniformly continuous function $h : I \to \mathbf{R}$ that is not *absolutely continuous.*

3. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the unit disk D?

4. Suppose λ denotes Lebesgue measure on $\mathbf{R}$, and $0 < C < 1$. Show that there are numbers $\delta_n \to 0$ (depending on C) with the following property: If $A_1, A_2, \ldots, A_n$ are measurable subsets of $[0,1]$ each with Lebesgue measure C, then $\lambda(A_i \cap A_j) \geq (1 - \delta_n)C^2$ for some $1 \leq i < j \leq n$.

Hint: Consider F^2 where $F = \chi_{A_1} + \chi_{A_2} + \ldots + \chi_{A_n}$.

Here χ_A is the *characteristic* (or *indicator*) *function*: $\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A. \end{cases}$

5. (a) Show that $\phi(z) = \frac{z-1}{z+1}$ maps the right half-plane $U = \{x + \mathbf{i}y \in \mathbf{C} \,:\, x > 0\}$ conformally onto the unit disk D.

(b) Find a conformal map onto the right half-plane U from the *left-slit-plane* $V = \mathbf{C} \setminus \{t \,:\, t \leq 0\} = \{x + \mathbf{i}y \in \mathbf{C} \,:\, y \neq 0 \text{ whenever } x \leq 0 \}$.

(c) Show that there exists a nonconstant bounded holomorphic function on V.

(d) Show that there exists a bounded harmonic function on V so that, for all $x < 0$, $\lim_{\pm y \downarrow 0} h(x + \mathbf{i}y) = \pm 1$.

6. (a) Define the space $L^1([0,1])$ and the L^1 norm on this space.

(b) Show that the function $F : [0,1] \to L^1(\mathbf{R})$,

$F(t) = \chi_{[0,t]}$ (the characteristic function of $[0,t]$)

satisfies $\|F(s) - F(t)\|_{L^1} \leq |s - t|$ for $0 < s < t < 1$.

(c) Show that, for any $0 < t < 1$, there does not exist a $g \in L^1([0,1])$ so that $\lim_{h \to 0} \|\frac{F(t+h) - F(t)}{h} - g\|_{L^1} = 0$. (i.e. F is not differentiable at t.)