Analysis Exam, August 2007

1. Suppose a and b are two points on the unit circle, and f is a nonconstant holomorphic function on the unit disk D.

(a) Show that

$$\lim_{t\downarrow 0} \frac{|f(ta)|}{|f(tb)|} = 1 .$$

(b) Is this still true if f is meromorphic with a pole at 0?
Prove if true or find a counterexample if false.

2. (a) Suppose $f_1, f_2, f_3, \cdots$ is a sequence of positive continuous functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(t)\,dt = \infty$. Is it possible that $\lim_{n\to\infty} f_n(t) = 0$ for all $0 \le t \le 1$?
If so, find an example. If not, explain.

(b) Suppose $g_1, g_2, g_3, \cdots$ is a sequence of positive continuous functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 g_n(t)\,dt = 0$. Show that, for almost every $t \in [0,1]$,

$$\lim_{n'\to\infty} g_{n'}(t) = 0 \text{ for some subsequence } n' \text{ of } n .$$

(c) In (b), is it possible that, for each $t \in [0,1]$, $\lim_{n''\to\infty} g_{n''}(t) = \infty$ for some subsequence n'' of n? If so, find an example. If not, explain.

3. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx$$

for any $t \in \mathbf{R}$.

4. Let λ denote the Lebesgue measure on $\mathbf{R}$, and suppose $C = \cap_{i=1}^{\infty} C_i$ where $C_1 \supset C_2 \supset C_3 \supset \cdots$ are closed subsets of $\mathbf{R}$.

(a) Show that $\lambda(C) \le \lim_{i\to\infty} \lambda(C_i)$.

(b) Is this always an equality? If so, explain. If not, find an example with strict inequality.

(c) Suppose in addition that $\lambda(C_1) = 1$ and $\lambda(C_i \setminus C_{i+1}) = \frac{1}{(i+1)^2}\lambda(C_i)$. Find $\lambda(C)$.

5. Suppose A is a finite subset of $\mathbf{C}$, f is a holomorphic function on $\mathbf{C} \setminus A$, $\lim_{z\to\infty} |f(z)| = +\infty$, and $\lim_{z\to a} |f(z)| = +\infty$ for all $a \in A$.
Prove that f is a rational function (that is, a quotient of 2 polynomials).

6. (a) Find an infinitely differentiable function f on the open interval $(-1,1)$ with $f(t) > 0$ for $0 < |t| < 1$ and all derivatives $f(0) = f'(0) = f''(0) = f^{(3)}(0) = \cdots = 0$.

(b) Give an example of two infinitely differentiable functions g and h on $(-1,1)$ which are not identically 0 but whose product $g \cdot h$ is identically zero.

(c) Show that this cannot happen if g and h are given by convergent power series $g(t) = \sum_{k=0}^{\infty} a_k t^k$ and $h(t) = \sum_{k=0}^{\infty} b_k t^k$. Hint: consider the functions $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} b_k z^k$ with z complex.