ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2014

Instructions:

- You have 4 hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 15th, 2014.

(1) Let p and q be prime numbers satisfying $p > q$, and let k be a non-negative integer.

(a) Prove that if H is a normal subgroup of order p^k of a finite group G, then H is contained in every Sylow p-subgroup of G.

(b) Prove that if G is a group of order $p^k q$, then G contains a unique normal subgroup of index q.

(2) Let p be a prime number, and let n be a positive integer.

(a) Show that the center of a finite group G of order p^n contains more than one element.

(b) Let $K/\mathbb{Q}$ be the splitting field of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$. Suppose that $[K : \mathbb{Q}] = p^n$. Prove that the extension $K/\mathbb{Q}$ is solvable.

(3) Let p be a prime integer, and let A be an $n \times n$ *integer* matrix such that $A^p = I$, but $A \neq I$ (here I denotes the identity matrix). Prove that $n \geq p - 1$. Give an example, with proof, that shows the lower bound $n = p - 1$ is attained.

(4) Let L/K be an extension of fields. An element $\alpha \in L$ is called primitive if $L = K(\alpha)$. Prove that a finite extension of finite fields has a primitive element.

(5) Let $R = \mathbb{Q}[x, y]$ denote the polynomial ring in two variables over the rational numbers.

(a) Let $I = \langle x, y \rangle$ denote the ideal generated by x and y, and $\phi \colon I \to R$ an R-linear homomorphism. Show that ϕ extends uniquely to an R-linear homomorphism $\phi' \colon R \to R$. In other words, if $j \colon I \to R$ is the inclusion, show there exists a unique $\phi' \colon R \to R$ such that $\phi' \circ j = \phi$.

(b) Does the same hold true for the ideal $I = \langle xy \rangle$?

(c) Let R' denote the localization of R at $\langle x, y \rangle$ and $\mathfrak{m}$ its maximal ideal. Compute the dimension of the quotient $\mathfrak{m}/\mathfrak{m}^2$ as a $\mathbb{Q}$-vector space.

(6) Let $\mathbb{Z}$ denote the ring of integers and α a complex number.

(a) Suppose there exists a monic polnomial $f(x) \in Z[x]$ such that $f(\alpha) = 0$. Show there exists a monic polynomial $g(x) \in Z[x]$ such that $g(\alpha^2) = 0$.

(b) Now suppose that $\alpha^2 + a_1\alpha + a_0 = 0$ for some integers a_1 and a_0. Find a monic polynomial $g(x) \in Z[x]$ such that $g(\alpha - 1) = 0$.

(c) Suppose that 2α and 3α arise as zeros of monic polynomials with integer coefficients. Is the same true for α?