Algebra Qualifying Exam

Rice University Mathematics Department

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You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. For $n \in \mathbb{N}$, let $(\mathbb{Z}/n\mathbb{Z})^*$ denote the group of invertible elements of the ring $\mathbb{Z}/n\mathbb{Z}$ and

$$M = \{n \in \mathbb{N} : (\mathbb{Z}/n\mathbb{Z})^* \text{ is a cyclic group}\}.$$

   Prove that the sets M and $\mathbb{N}\backslash M$ are both infinite.

2. Let V be a finite rank free module over a PID R, W be a submodule of V, and $\eta : V \to V/W$ be the natural projection to the quotient module V/W. Show the following are equivalent:

   (i) There is a R-homomorphism $\lambda : V/W \to V$ such that $\eta \circ \lambda = id_{V/W}$.

   (ii) V/W is a free R-module.

   (iii) $V = ker(\eta) \oplus P$ where $\eta|P : P \to V/W$ is an isomorphism.

3. Let K be a Galois extension of a field F and L be an intermediate field, $F \subseteq L \subseteq K$. Put $H = G(K/L) \subseteq G(K/F) = G$.

   (a) Show that $\{\sigma \in G : \sigma(L) = L\}$ is the normalizer $N(H) = \{g \in G : gHg^{-1} = H\}$ of H in G. Deduce that $N(H) = H$ if and only if every $\sigma \in G$ which maps L to itself is the identity on L.

(b) Let K be the splitting field of an irreducible $f(x) \in \mathbb{Q}[x]$ with exactly one real root. Show that $G(K/\mathbb{Q})$ has a subgroup of index $n = deg(f(x))$ which is not normal in any larger subgroup.

4. Let $x_n = \cos n^\circ$ and $y_n = \sin n^\circ$, for $n \in \mathbb{N}$. Here, we express angles in terms of degrees, so that $\cos 90^\circ = 0$. Prove:

   (a) Both x_n and y_n are algebraic numbers, for all $n \in \mathbb{N}$.

   (b) Both $2x_n$ and $2y_n$ are algebraic integers, for all $n \in \mathbb{N}$.

   (c) x_n is an algebraic integer if and only if y_n is.

   (d) x_1 fails to be an algebraic integer.

5. (a) Let $f(x, y), g(x, y) \in \mathbb{C}[x, y]$ denote polynomials with no common roots in $\mathbb{C}^2$. Show there exist polynomials $P(x, y), Q(x, y) \in \mathbb{C}[x, y]$ such that

$$Pf + Qg = 1.$$

   (b) Find an example where this fails over $\mathbb{R}$, i.e., $f(x, y), g(x, y) \in \mathbb{R}[x, y]$ with no common roots in $\mathbb{R}^2$, for which there exist no polynomials P and Q with $Pf + Qg = 1$.

6. Consider the set of matrices

$$\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}/5\mathbb{Z}\},$$

   which is a group under matrix multiplication.

   (a) Show that $|\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})| = 120$.

   (b) Show that

$$B = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : \quad b \in \mathbb{Z}/5\mathbb{Z}\} \subset \mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$$

   is a Sylow subgroup, and $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ admits six 5-Sylow subgroups, in total.