Algebra Qualifying Exam

Rice University Mathematics Department

January 10, 2010

You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. Let $p, q \geq 2$ be primes and assume that $q|(2^p - 1)$. Prove that $p|(q-1)$. (We write $x|y$ to denote the fact that x divides y, i.e., that y/x is an integer.) *Hint:* Think in terms of groups.

2. Let $M = \begin{pmatrix} 7 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{pmatrix}$. Find the following:
   - the characteristic polynomial of M;
   - the minimal polynomial of M;
   - the Jordan canonical form for M.

3. Prove that a group of order 105 cannot be simple. Must such a group be abelian?

4. Consider the ideal $I = \langle x^2, y \rangle \subset R = \mathbb{Q}[x, y]$.
   - Compute $\mathrm{Tor}_n^R(I, R/I)$ and $\mathrm{Ext}_R^n(I, R/I)$ for each $n \geq 0$.
   - Show that the R-module $I \quad {}_R I$ cannot be realized as an ideal in R.
   - Let P denote the prime ideal $\langle x-1, y-2 \rangle$. Show that the localization of I at P is a free R_P module.

5. Let $K = \mathbb{Q}(e^{\frac{2\pi i}{5}}) \subset \mathbb{C}$. Show that

   - $i = \sqrt{-1} \notin K$;
   - $K \cap \mathbb{R} = \mathbb{Q}(\sqrt{5})$.

6. Let R be a commutative ring with 1, M an R-module, and $N \subset M$ an R-submodule.

   - Assume R is Noetherian. Show that N is finitely generated if M is finitely generated.
   - Assume R is a principal ideal domain and M can be generated by m elements. Show N can be generated by m elements.
   - Suppose R is a unique factorization domain and M is generated by m elements. Is N necessarily generated by m elements?