ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2017

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: May 9, 2017.

(1) Classify, up to isomorphism, groups of order 21.

(2) (a) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible, has degree 4, and has exactly two real roots. What are the possibilities for the Galois group of the splitting field of $f(x)$? Justify.

(b) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible, has degree 3, and has negative discriminant. What are the possibilities for the Galois group of the splitting field of $f(x)$? Justify.

(3) Let i denote a primitive fourth root of unity.

(a) Determine the irreducible polynomial for $a = i + \sqrt{2}$ over $\mathbb{Q}$.

(b) Determine the automorphism group of $\mathbb{Q}(a)$. Is $\mathbb{Q}(a)$ a Galois extension of $\mathbb{Q}$?

(4) Let $M_2(\mathbb{C})$ denote the complex vector space of 2×2 matrices. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $T\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be the linear transformation defined by

$$T(X) = XA - AX.$$

Determine the Jordan canonical form for T.

(5) Let A be a commutative ring with unit, and let $S \subset A$ be a multiplicatively closed subset.

(a) Suppose that A is Noetherian. Is the localization $S^{-1}A$ Noetherian?

(b) Suppose instead that $S^{-1}A$ is Noetherian. Must the original ring A be Noetherian?

(6) Let $f\colon A \to B$ be an injective map between commutative rings with unit that makes B an integral extension of A. Prove that the induced map on prime spectra $f^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed, i.e., it maps closed sets to closed sets.

[Hint: First reduce the problem to proving that $f^*(\mathrm{Spec}(B))$ is closed in $\mathrm{Spec}(A)$.]