Algebra Qualifying Exam

Rice University Mathematics Department

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You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. For R a ring R^* denotes the multiplicative group of units in R; C_n denotes the cyclic group of order n.

   a. Show that $(\mathbb{Z}/(n))^* \cong \mathrm{Aut}(C_n)$.

   b. Show that for finite groups A and B of relatively prime order $((o(A), o(B)) = 1)$ we have $\mathrm{Aut}(A \times B) \cong \mathrm{Aut}(A) \times \mathrm{Aut}(B)$.

   c. Describe $(\mathbb{Z}/(35))^*$ as a product of cyclic groups.

2. For a permutation $\sigma \in S_n$ (the symmetric group on n letters), let T_σ denote the corresponding permutation matrix which takes e_i to e_{i^σ} for $i = 1, 2, \ldots, n$. Here $e_1, \ldots, e_n$ is a basis for $\mathbb{R}^n$.

   a. Explain how the decomposition of σ as a product of disjoint cycles gives a decomposition of $\mathbb{R}^n$ as a direct sum of T_σ-invariant subspaces.

   b. Determine the characteristic polynomial of T_σ in terms of the data in part (a).

   c. Show that $\sigma \in A_n$ (the alternating group) if and only if $\det(T_\sigma) = 1$.

3. Suppose V is a vector space over $\mathbb{Q}$ with basis $\{x, y, z\}$.

a. Find a **vector space** basis for the (entire) alternating (exterior) algebra $\wedge^*(V)$.

b. Find an **algebra basis** (set of minimal cardinality that generates as an algebra) for $\wedge^*(V)$.

c. Find a **vector space** basis for $\oplus_{p=0}^{2} S^p(V)$, where $S^p(V)$ denotes the subspace of degree p homogeneous elements of the symmetric algebra of V.

4. Let R be a commutative domain.

   a. Prove: if an R module M is projective, then it is flat.

   b. Prove: if an R module M is flat, then it is torsion-free.

   In your proofs, the definitions of the terms projective, flat and torsion-free should be explicitly given.

5. Suppose that R is a commutative Noetherian ring and $S \subset R$ is a multiplicative subset. Prove that $S^{-1}R$ is a Noetherian ring.

6. A field $K \subset \mathbb{C}$ is called FPIN (free of purely imaginary numbers) if

$$K \cap \{ri \mid r \in \mathbb{R}\} = \{0\}.$$

   a. Prove that if $K \subset \mathbb{C}$ is the Galois field of a polynomial $f \in \mathbb{Q}[x]$ then K is FPIN if and only if $K \subset \mathbb{R}$.

   b. Let $L \subset \mathbb{C}$ denote the splitting field of $x^3 - 2 \in \mathbb{Q}[x]$. Which intermediate fields

$$\mathbb{Q} \subsetneq K \subsetneq L$$

   are FPIN? Are they all contained in $\mathbb{R}$?