# Algebra Qualifying Exam

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1. Show that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ has $p+1$ subgroups of order $p$, when $p$ is prime. Show that the group of automorphisms of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $S_3$ by considering its action on the subgroups of order 2.

2. Let $f(x) = x^8 + 1$.

   a. Let $K$ be the splitting field of $f(x)$ over $\mathbb{Q}$, the field of rational numbers. Determine the Galois group of $K/\mathbb{Q}$.

   b. How many subfields of $K$ are of degree 4 over $\mathbb{Q}$? How many of these are Galois over $\mathbb{Q}$? Explain.

   c. Let $L$ be the splitting field of $f(x)$ over $\mathbb{F}_{41}$, the field of 41 elements. Determine the Galois group of $L/\mathbb{F}_{41}$.

3. Let $p$ be a prime. Show that each group of order $p^2$ is abelian. Give an example of a non-abelian group of order $p^3$ for each prime $p$.

4. Let $R$ be a PID and $F$ its field of fractions. Suppose $S$ is a ring with $R \subset S \subset F$.

   a. Show that all elements $\alpha \in S$ can be written as $a/b$, where $a$, $b \in R$ and $1/b \in S$.

   b. Show that $S$ is a PID.

   c. Show that if $S$ is finitely generated as an $R$-module then $S = R$.

5. Let $V = \mathbb{R}^n$ and consider two sets of linearly independent vectors $\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\} \subset V$.

   a. Show that $v_1 \wedge v_2 \wedge v_3 = cw_1 \wedge w_2 \wedge w_3 \in \bigwedge^3 V$ for some $c \in \mathbb{R}$ if and only if $\text{span}(v_1, v_2, v_3) = \text{span}(w_1, w_2, w_3)$.

   b. Does $\text{span}(v_1, v_2, v_3) = \text{span}(w_1, w_2, w_3)$ imply that $v_1 \quad v_2 \quad v_3 = cw_1 \quad w_2 \quad w_3 \in \mathcal{T}^3 V$, for some $c \in \mathbb{R}$?

6. Let $R = \mathbb{Q}[x, y]$ and $I = \langle x, y \rangle \subset R$ the ideal generated by $x$ and $y$. Show that $R/I$ is neither flat nor projective as an $R$-module. Do the same for $I$.