ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2015

Instructions:

- You have 4 hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 21st, 2015.

(1) Let F_n be the free group on $\{x_1, \dots, x_n\}$, and let $(\mathbb{Z}/2\mathbb{Z})^n = \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$.

(a) Denote by K the kernel of the surjection ϕ that maps

$$x_i \mapsto (0, \dots 0, 1, 0, \dots 0), \qquad i = 1, \dots, n$$

with 1 being the ith entry. Show that any automorphism of F_n preserves K. Prove that any automorphism of F_n induces an automorphism of $(\mathbb{Z}/2\mathbb{Z})^n$ via ϕ.

(b) For $n = 3$, determine the automorphism $f\colon (\mathbb{Z}/2\mathbb{Z})^3 \to (\mathbb{Z}/2\mathbb{Z})^3$, as a 3×3 matrix, induced by the automorphism $x_1 \mapsto x_1 x_3 {x_2}^2 {x_1}^{-1}$, $x_2 \mapsto x_1 x_3 x_2$, $x_3 \mapsto x_3 x_2$ of F_n.

(2) Consider the polynomials $f(x) = x^2 + x + 2$ and $g(x) = x^2 + 2x + 2$ in $\mathbb{F}_3[x]$.

(a) Show that both f and g are irreducible.

(b) Are the fields $\mathbb{F}_3[x]/(f(x))$ and $\mathbb{F}_3[x]/(g(x))$ isomorphic? If so, exhibit an isomorphism between them. If not, prove so. Justify your claims.

(3) Let $a = \cos(2\pi/9)$.

(a) Compute the minimal polynomial $P(x)$ of a over $\mathbb{Q}$.

(b) Is $\mathbb{Q}(a)/\mathbb{Q}$ separable? Is it a splitting field for $P(x)$?

(c) Compute $\mathrm{Aut}(\mathbb{Q}(a)/\mathbb{Q})$.

Carefully justify your answers.

(4) Let R be a ring and let M be an R-module. The support of M is the set

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R : M_{\mathfrak{p}} \neq 0\},$$

where $M_{\mathfrak{p}}$ denotes the localization of M at $\mathfrak{p}$.

(a) Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Prove that

$$\mathrm{Supp}(L) \cup \mathrm{Supp}(N) = \mathrm{Supp}(M).$$

(b) Let $\{M_i\}$ be a collection of submodules of M with $M = \sum_i M_i$. Prove that

$$\mathrm{Supp}(M) = \bigcup_i \mathrm{Supp}(M_i).$$

[Hint: Consider the map $\bigoplus_i M_i \to M$.]

(c) Let M and N be R-modules. Show that $\mathrm{Supp}(M \otimes_R N) \subseteq \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.

(5) Find fields K_1 and K_2 such that

$$\mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{3}) \simeq K_1 \times K_2.$$

Carefully justify your answer.

(6) Consider the ideal $I = (t^2 + t - x, t - 1 - y)$ in $\mathbb{Q}[t, x, y]$.

(a) Show that $\{t - y - 1, x - y^2 - 3y - 2\}$ is a Gröbner basis for I for the lexicographic order $t > x > y$.

(b) Compute a set of generators for the kernel of the ring homomorphism $\mathbb{Q}[x, y] \to \mathbb{Q}[t]$ given by

$$x \mapsto t^2 + t, \qquad y \mapsto t - 1.$$