ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2014

Instructions:

- You have 3 hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

(1) Let G be a finite group of odd order.

(a) Show that any subgroup $H \subset G$ of index three is normal.

(b) Show by example that a subgroup of index five need not be normal.

(2) Are any of the rings $\mathbb{Z}[\sqrt{5}]/(2)$, $\mathbb{Z}[\sqrt{-2}]/(2)$, and $\mathbb{Z}[\sqrt{-3}]/(1+\sqrt{-3})$ isomorphic to one another? Justify.

(3) Suppose that $K/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Prove that $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where d_1, $d_2 \in \mathbb{Q}$ have the property that none of d_1, d_2, or $d_1 d_2$ is a square in $\mathbb{Q}$.

(4) Let V be a finite-dimensional vector space over a field of characteristic 0. Suppose that $T\colon V \to V$ is a linear operator such that the trace of T^k is 0 for all integers $k > 0$.

(a) Prove that the determinant of T is 0, and conclude that T is not surjective (hint : apply Cayley-Hamilton).

(b) Set $W = T(V)$, so W is a proper subspace of V which is preserved by T. Define $S\colon W \to W$ to be the restriction of T to W. Prove that the trace of S^k is 0 for all integers $k > 0$.

(c) Prove that T is nilpotent, i.e., that $T^p = 0$ for some integer $p > 0$.

(5) Let $R = \mathbb{Q}[x, y]$ and let M and N be finitely generated modules over R.

(a) Show that if M and N are flat then $M \otimes_R N$ is flat as well.

(b) If M and N are projective does it follow that $M \otimes N$ is projective as well?

(6) Let R be a principal ideal domain, and let Q be an R-module.

(a) Show that Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$.

(b) Can a nonzero finitely generated abelian group, considered as a $\mathbb{Z}$-module, be injective?

Date: August 21st, 2014.