Algebra Qualifying Exam

Rice University Mathematics Department

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You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. Describe, with proof, four mutually non-isomorphic groups of order 50. In particular, construct the groups clearly and show carefully that the resulting groups are non-isomorphic.

2. Let $\Phi_{90}(x)$ denote the monic polynomial whose roots are the primitive 90th roots of unity; it is irreducible.

   a. Show that $\Phi_{90}(x) \in \mathbb{Z}[x]$, i.e., the coefficients are integers.

   b. Determine the splitting field of Φ_{90} as a polynomial over the finite field $\mathbb{F}_{11} = \mathbb{Z}/11\mathbb{Z}$.

   c. Now regard Φ_{90} as a polynomial over $\mathbb{Q}$. Describe, in detail, its Galois group.

3. Let R be an integral domain. Assume that

   - $ab = cd$ holds, for some $a, b, c, d \in R$;
   - a and b are prime elements in R.

   Prove or disprove: The element c must be an associate of one of the following elements: $a, b, ab, 1_R$ (the identity in R).

4. Let A be a real 9×9 matrix with transpose B. Prove that the matrices A and B are real equivalent in the following sense: There exists a real

invertible 9×9 matrix H such that $AH = HB$. *For partial credit*: Establish the existence of a *complex* invertible matrix H with $AH = HB$.

5. Consider the rings

$$R := \mathbb{Z}[\sqrt{-3}] \subset S := \mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \subset \mathbb{C};$$

regard S as an R module.

a. Show that S is finitely generated as an R-module.

b. Let $\mathfrak{p} \neq 0$ be a prime ideal of R and consider the localizations

$$R_{\mathfrak{p}} \subset S_{\mathfrak{p}}.$$

Show these are equal if $\mathfrak{p}$ does not contain 2.

c. Show that S is neither flat nor projective as an R module.

6. Let e_1, e_2, e_3, e_4 be a basis for $\mathbb{R}^4$ and

$$q = e_1e_2 + e_3e_4 \in \mathrm{Sym}^2(\mathbb{R}^4),$$

i.e., an element of the symmetric algebra $\mathrm{Sym}(\mathbb{R}^4)$. Show there do not exist elements $v, w \in \mathbb{R}^4$ such that $q = vw$ in $\mathrm{Sym}(\mathbb{R}^4)$.