RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2018

This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Prove that $SL_n(\mathbb{R})$ is a smooth manifold and calculate its dimension (with proof).

2. Let $X = S^1 \times \mathbb{R}P^2$, where $\mathbb{R}P^2$ is the real projective plane. Discuss all of the covering spaces of X. How many are there? Describe them. What are their groups of covering transformations? Which ones are regular (normal)?

3. Let $F(n)$ be the free group of rank n. For each integer $n \geq 2$, prove that $F(2)$ contains a finite index normal subgroup isomorphic to $F(n)$.

4. Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. That is,
$$S(X) = X \times [0,1]/\sim$$
where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$. Describe the relation between the cohomology groups of X and $S(X)$.

5. Let X be a closed, oriented 4-manifold with $\pi_1(X) \cong \mathbb{Z}/15\mathbb{Z}$ and $\chi(X) = 5$. Let $\widetilde{X}$ be a connected 3-fold covering space of X. Calculate $H_i(\widetilde{X};\mathbb{Z})$ for all i.

6. Let X be a closed (compact and boundaryless), oriented 4-manifold with $\beta_2(X) \neq 0$. Prove that any continuous map $f : S^4 \to X$ has degree equal to 0.