RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2018

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Give an example (a path connected CW complex) for each of the following or state that such an example does not exist. Give a brief justification in all cases.
   (a) Two spaces with isomorphic π_1 but non-isomorphic integral homology groups.
   (b) Two spaces with isomorphic integral homology groups but non-isomorphic π_1 (give π_1 of the spaces).
   (c) Two spaces that are homotopy equivalent but not homeomorphic.
   (d) Two spaces with isomorphic π_1 and isomorphic integral homology groups that are NOT homotopy equivalent.

2. Let $W = S^1 \vee S^1$ be the wedge of two circles (see the figure below).

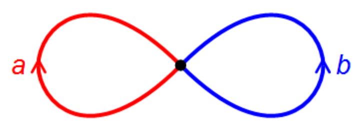

Figure 1. Wedge of two circles

   (a) Let $S = \langle a \rangle$ be the subgroup of $\pi_1(W)$ generated by a. Describe the covering space corresponding to S. Is this covering space regular? What is its group of deck transformations?
   (b) Let C be the commutator subgroup of $\pi_1(W)$. Recall that C is the subgroup of $\pi_1(W)$ generated by $xyx^{-1}y^{-1}$ where $x, y \in \pi_1(W)$. Describe the covering space corresponding to C. Is this covering space regular? What is its group of deck transformations?

3. Let H be a solid handlebody of genus 2. Recall that H can be obtained as follows. Let S be the surface (with boundary) pictured in Figure 2, then $H = S \times I$. H can also be viewed as a 3-dimensional manifold obtained by thickening up a wedge of circles (note that the boundary of H is a genus 2 surface).

   Let $f : H \to \mathbb{R}^3$ be a topological embedding. Let $X = \text{int}(f(H))$ be the image of f in $\mathbb{R}^3$. Compute $H_p(\mathbb{R}^3 - X)$ for all p.

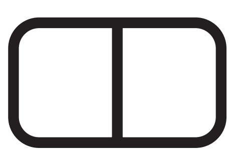

Figure 2. The surface S

4. Let X be a compact, connected, orientable 4-dimensional manifold without boundary such that $\pi_1(X) \cong \mathbb{Z}_{15}$ and $H_2(X;\mathbb{Q}) \cong \mathbb{Q}^2$. Let E be a connected 3-fold covering space of X.
   (a) Calculate $\pi_1(E)$.
   (b) Calculate $H_p(X;\mathbb{Z})$ for each p.
   (c) Calculate $\chi(X)$.
   (d) Calculate $H_p(E;\mathbb{Z})$ for each p.
   (e) Prove that E admits no CW decomposition without 3-cells.

5. Let X and Y be compact, connected, oriented n-dimensional manifolds without boundary and let $f : X \to Y$ be a continuous map. Suppose $\beta_p(X) < \beta_p(Y)$ for some $p > 0$.
   (a) Prove that $f^* : H^p(Y;\mathbb{Q}) \to H^p(X;\mathbb{Q})$ has a non-trivial kernel.
   (b) Show that f is a degree zero map.

6. Let S^3 be the unit 3-sphere. Prove that the tangent bundle to S^3 is trivial

$$TS^3 \cong S^3 \times \mathbb{R}^3$$

by exhibiting three explicit linearly independent vector fields X_1, X_2, X_3 (on S^3).