# RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2008

This is a 3-hour closed book, closed notes exam. Please show all of your work.

1. Let $W = S^1 \vee S^1$ be the wedge of 2 circles. Describe four distinct connected 3-fold covering spaces of $W$ including at least one irregular cover. In each case, give the group of covering transformations, say whether or not the covering is normal (regular) and give the corresponding subgroup of $\pi_1(W)$.

2. Give an example (a CW complex) for each of the following or state that such an example does not exist. Give a brief justification in all cases.

(a) Two spaces with isomorphic $\pi_1$ but non-isomorphic integral homology groups. (b) Two spaces with isomorphic integral homology groups but non-isomorphic $pi_1$ (give $\pi_1$ of the spaces). (c) Two spaces with isomorphic integral homology groups but non-isomorphic cohomology groups. (d) Two spaces that are homotopy equivalent but not homeomorphic. (e) Two spaces with isomorphic $\pi_1$ and isomorphic integral homology groups that are NOT homotopy equivalent.

3. Let S be the closed surface obtained as the connect sum of 3 projective planes.
(a) Construct an explicit orientable 2-fold cover of S (with covering map).
(b) For a given integer d classify all d fold coverings of S up to homeomorphism.

4. (a) State the Mayer-Vietoris theorem
(b) Let X be a topological space. Define the suspension S(X) to be the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and S(X).

5. Suppose $f : I \to Y$ is a path in $Y$ and $g : I \to I$ is a continuous map such that $g(0) = 0$ and $g(1) = 1$. Prove that $f$ is homotopic to $f \circ g$.

6. Let $X$ and $Y$ be compact, connected, oriented $n$-dimensional manifolds without boundary and let $f : X \to Y$ be a continuous map. Suppose $\beta_p(X) < \beta_p(Y)$ for some $p > 0$.

a) Prove that $f^* : H^p(Y; \mathbb{Q}) \to H^p(X; \mathbb{Q})$ has a non-trivial kernel.
b) Show that $f$ is a degree zero map.