Analysis Exam, May 2019, 4 hours

Please put your name on your solutions, use $8\frac{1}{2} \times 11$ in. sheets, and number the pages.

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of entire functions and f an entire function such that f_n converges to f uniformly on $\{z \in \mathbb{C} \mid |z| \leq 1\}$. Prove that $\lim_{n\to\infty} f_n'(0) = f'(0)$.

2. Let λ_n denote n dimensional Lebesgue measure on $\mathbb{R}^n$. Suppose that, for each $j = 1, 2, \dots$, A_j is a Borel subset of $[0,1] \times [0,1]$ with $\lambda_2(A_j) > \frac{1}{3}$, and

$$B_j = \left\{ x \in [0,1] \; : \; \lambda_1 \left(\{ y \; : \; (x,y) \in A_j \} \right) > \frac{1}{4} \right\}.$$

   (a) Prove that $\lambda_1(B_j) > \frac{1}{9}$ for every j.

   (b) Prove that $\lambda_1 \left(\{ x \; : \; x \in \text{infinitely many } B_j \} \right) \geq \frac{1}{9}$. (Hint: Fatou's Lemma)

3. Suppose $U = \{z \in \mathbb{C} \; : \; \mathcal{I}m(z) > [\mathcal{R}e(z)]^2\}$. Prove or disprove:

   (a) There exists a holomorphic map from U onto $\mathbb{C}$.

   (b) There exists a holomorphic map from $\mathbb{C}$ onto U.

   (c) There exists a holomorphic map from $U \setminus \{i\}$ onto $\mathbb{C} \setminus \{0\}$.

   (d) There exists a holomorphic map from $\mathbb{C} \setminus \{0\}$ onto $U \setminus \{i\}$.

4. Suppose $1 \leq p < q < r < \infty$.

   (a) Show that $L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subset L^q(\mathbb{R})$.

   (b) Show that $L^q(\mathbb{R}) \subset L^p(\mathbb{R}) + L^r(\mathbb{R})$ (we use the notation $A + B = \{g + h : g \in A, h \in B\}$)

5. Let n be a positive integer and let $a \in (0,1)$. Compute

$$\int_0^{2\pi} \frac{\cos nx}{1 - 2a\cos x + a^2} \, dx.$$

6. Suppose f and $f_1, f_2, f_3, \dots$ belong to $L^1([0,1])$ and $f_n(x) \to f(x)$ for a.e. $x \in [0,1]$. Show that, for every $\varepsilon > 0$, there is a compact $K \subset [0,1]$ with Lebesgue measure $> 1 - \varepsilon$ so that

$$\lim_{n\to\infty} \int_K |f_n - f| \, dx \; = \; 0 \; .$$