Analysis Exam, May 2018

Please put your name on your solutions, use 8 1/2 x 11 in. sheets, and number the pages.

1. On the open unit ball $\mathbf{B}$ in the complex plane $\mathbf{C}$, find all holomorphic functions h satisfying

$$h(z)^2 \;=\; h(z) \quad \text{for all } \; z \in \mathbf{B}\ .$$

2. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of real numbers.

(a) Does the convergence of the series $\sum_{n=1}^{\infty} a_n^3$ imply the convergence of $\sum_{n=1}^{\infty} \frac{a_n}{n}$? If so, prove this. If not, give a counterexample.

(b) Does the convergence of $\sum_{n=1}^{\infty} \frac{a_n}{n}$ imply the convergence of $\sum_{n=1}^{\infty} a_n^3$? If so, prove this. If not, give a counterexample.

3. Let $u(x,y)$ be harmonic on a connected domain $D \subset \mathbf{C}$ and let $v(x,y)$ be a harmonic conjugate of $u(x,y)$ on D.

(a) Prove that $u(x,y)v(x,y)$ is harmonic on D.

(b) Prove that if $xu(x,y)$ is harmonic on D, then $u(x,y) = ay + b$, where a and b are constants.

4. (a) Define the total variation of a function $f : [0,1] \to \mathbf{R}$.

(b) Assuming f has finite total variation, estimate the total variation of the function $g : [0,1] \to \mathbf{R}$, $g(x) \;=\; \int_0^1 f(xy)\, dy$, in terms of the total variation of f.

(c) Show that g is absolutely continuous whenever f is absolutely continuous.

5. (a) Suppose N is a positive integer and f is a not-identically-zero meromorphic function on all of $\mathbf{C}$ satisfying

$$\limsup_{|z|\to\infty} \frac{|f(z)|}{|z|^N} \;<\; N\ .$$

Does f have only a finite number of poles?

If so, prove this. If not, give an example with an infinite number of poles.

(b) Does f have only a finite number of zeroes? If so, prove this. If not, give an example with an infinite number of zeroes.

6. Let f_n be a sequence of increasing differentiable functions on $\mathbf{R}$ such that, at each point $x \in [0,1]$, we have the convergence of the infinite series

$$F(x) \;=\; \sum_{n=1}^{\infty} f_n(x) \;=\; \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x)\ .$$

Since F is an increasing function, you may assume that F is differentiable a.e. (*almost everywhere*)

(a) Give an example (or sketch the graphs) of such a sequence f_n whose sum F is discontinuous at 0.

(b) Prove that if g_n are increasing differentiable functions on $\mathbf{R}$ with convergent series $G(x) \;=\; \sum_{n=1}^{\infty} g_n(x)$, then

$$G' \;\geq\; \sum_{n=1}^{\infty} g_n' \text{ a.e. on } [0,1]\ .$$

(c) Prove that $G' \;\leq\; \sum_{n=1}^{\infty} g_n'$ a.e. on $[0,1]$.