ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2017

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 10, 2017.

(1) Consider the following matrices

$$M_1 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \text{ and } M_2 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

Are the $\mathbb{Z}$-modules $R_1 = \mathbb{Z}^2/M_1 \cdot \mathbb{Z}^2$ and $R_2 = \mathbb{Z}^2/M_2 \cdot \mathbb{Z}^2$ isomorphic? If so, explain why and describe an isomorphism. If not, explain why not.

(2) Let K/F be a field extension, and $\alpha \in K$ an element which is algebraic and of odd degree over F. Show that $F(\alpha^2) = F(\alpha)$.

(3) Give an example of a pair of square matrices A, B with $\mathbb{C}$-coefficients such that the minimal polynomials $m_A(x)$ and $m_B(x)$ are equal, the characteristic polynomials $c_A(x)$ and $c_B(x)$ are equal, but the matrices A and B are *not* conjugate.

(4) Let R be a ring with a unique prime ideal and $\mathrm{nil}(R) = (0)$. Prove that R is a field. [Here $\mathrm{nil}(R)$ stands for the nilradical of R]

(5) Let $p \in \mathbb{Z}_{>0}$ be a prime and $R = \mathbb{F}_p[x] \otimes_{\mathbb{F}_p[x^p]} \mathbb{F}_p[x]$.

(a) Prove that $x \otimes 1 - 1 \otimes x \in \mathrm{nil}(R)$.

(b) Prove that $\mathrm{nil}(R) = (x \otimes 1 - 1 \otimes x)$. [Hint: Consider the ring $R/(x \otimes 1 - 1 \otimes x)$.]

(6) Let ζ be a primitive 7^{th} root of unity, considered as a complex number, and let $F = \mathbb{Q}(\zeta)$ be the extension of the rational numbers obtained by adjoining ζ. Find, with proof, a primitive element that generates a subfield of F of degree 2 over $\mathbb{Q}$.