Rice University Algebra Qualifying Exam Syllabus

Group theory.

1. Basic theory : subgroups, normal subgroups, cosets, quotients, isomorphism theorems, conjugacy classes, direct products, semidirect products, commutator subgroup, solvable and nilpotent groups.

2. Examples : abelian groups, cyclic groups, dihedral groups, symmetric groups, alternating groups, finite matrix groups.

3. Group actions : definition, orbits, stabilizers, orbit-stabilizer theorem, conjugation action, action on cosets, symmetry groups of simple geometric objects (e.g. regular polygons).

4. Generators and relations : free groups, definition of group presentation, basic examples (eg surface groups).

5. Finite group theory : Sylow theorems, classification of groups of small order.

References : A standard reference for this is Dummit–Foote's *Abstract Algebra*, chapters 1–6. A somewhat easier reference that covers some of it is Artin's *Algebra*, chapters 2,6,7; however, this does not include everything. A more advanced but still very readable source is Alperin–Bell's *Groups and Representations*.

Elementary Ring and Module theory.

1. Ring Theory : ideals, homomorphisms, quotient rings and their ideals, product rings. Maximal and prime ideals.

2. Polynomial rings : division algorithm, Gröbner bases and applications to the ideal membership problem and elimination theory. Hilbert Basis Theorem.

3. Factorization : irreducible vs prime elements, Euclidean domains, principal ideal domains and unique factorization domains. Gauss' Lemma; factorization of polynomials in $\mathbb{Z}[x]$ (Eisenstein's criterion, reduction mod p).

4. Module Theory : submodules, quotient modules, free modules, finitely presented modules and presentation matrices. Smith normal form of a matrix over a PID.

5. Structure theory : finitely generated modules over a PID, with an emphasis on finitely generated abelian groups. Subgroups of finitely generated free abelian groups. Applications to linear algebra: cyclic modules over $F[t]$ (F a field); Jordan and rational canonical forms of matrices and their computation.

References: Artin's *Algebra* (second edition), chapters 11, 12 and 14. Dummit–Foote's *Abstract Algebra* has good treatments of Gröbner bases (section 9.6), and Jordan and rational canonical forms (chapter 12).

Fields and Galois Theory.

1. Basic Theory : algebraic and transcendental elements. Degree of a field extension. Adjoining roots of polynomials. The primitive element theorem.

2. Finite fields : existence of a field $\mathbb{F}_q$ of cardinality $q = p^r$, p prime. The elements of $\mathbb{F}_q$ are the roots of $x^q - x$; uniqueness of $\mathbb{F}_q$ up to isomorphism. The group $\mathbb{F}_q^\times$ is cyclic. Subfields of finite fields.

3. Isomorphisms of field extensions : finite Galois groups and Galois extensions, splitting fields. Action of the Galois group on the roots of a polynomial that splits completely in a Galois extension.

4. The Main Theorem of Galois Theory (i.e., the inclusion reversing correspondence between intermediate fields of a finite Galois extension and subgroups of its Galois group).

5. The Galois group of a polynomial : Galois groups of quadratic, cubic, quartic and cyclotomic polynomials. Solvability in radicals.

References: Artin's *Algebra* (second edition), chapters 15 and 16.

Commutative algebra.

1. Further ideal theory : nilradical, Jacobson radical, radical of an ideal, ideal quotients and the annihilator of an ideal, extension and contraction of ideals.

2. Further module theory : direct sums and products, direct and inverse limits; universal mapping properties. Complexes and exact sequences.

3. Multilinear Algebra : tensor, exterior and symmetric algebras over rings, with an emphasis on fields. Free, flat and projective modules.

4. Local rings and localization of modules : exactness of localization and local properties of modules (e.g., being zero or flat) and module homomorphisms (e.g., being injective or surjective).

5. Chain conditions and composition series : Noetherian and Artinian modules and rings.

6. Integral extensions : the going-up theorem, integral closure of an integral domain. Ring of integers of a number field.

7. Affine algebraic geometry : Hilbert's Nullstellensatz and the correspondence between algebraic subsets of $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$. The Zariski topology on the prime spectrum of a ring.

References : Atiyah–Macdonald's *Introduction to Commutative Algebra*, chapters 1, 2, 3, 5 (pp. 59–63), 6, 7 (pp. 80–82), 8. Reid's *Undergraduate Commutative Algebra* has a particularly good treatment of the Nullstellensatz and affine algebraic geometry (chapters 4 and 5). Dummit–Foote's *Abstract Algebra* has a lot of great examples and exercises.