ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2019

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: May 8th, 2019.

(1) Let p be an odd prime number.

(a) Prove that $\# \operatorname{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) = (p^2-1)(p^2-p)$.

(b) Prove that there exist non-abelian groups of order p^3.

(2) Compute the Galois group of the splitting field of each of the following polynomials over $\mathbb{Q}$. Justify your answers.

(a) $x^3 + 4x - 1$;

(b) $x^4 - x - 1$.

(3) In this problem, you may assume that the polynomials $x^2 - 4x + 1$ and $x^2 + 5x + 1$ are irreducible in $\mathbb{F}_{19}[x]$. Determine all similarity classes of 2×2 matrices M over $\mathbb{F}_{19}$ such that $M^5 = \mathrm{Id}$.

(4) Find fields K_1 and K_2 such that

$$\mathbb{Z}/11\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{3}] \simeq K_1 \times K_2,$$

as rings. Be sure to carefully justify your answer.

(5) Let A be a commutative ring with unit, and let N, N' be submodules of an A-module M.

(a) Show that $N' \subseteq N$ if and only if $N'/(N \cap N') = 0$.

(b) Prove that if $N_{\mathfrak{m}} = N'_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset A$, then $N = N'$.

(6) Let A be a commutative ring with unit. Let (M_i, μ_{ij}) be a directed system of A-modules over a directed set I, with direct limit M. Let N be an A-module. Then $(M_i \otimes_A N, \mu_{ij} \otimes \mathrm{id}_N)$ is also a directed system. Show that

$$\varinjlim (M_i \otimes_A N) \simeq \left(\varinjlim M_i\right) \otimes_A N$$

as A-modules.